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Canonical quadratic refinements of cohomological pairings from functorial lifts of the Wu class

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Abstract

We consider categories of manifolds admitting a functorial lift of their Wu class to integral cohomology. We show that the existence of a functorial lift allows to construct canonical quadratic refinements of various pairings defined on the cohomologies of a manifold of dimension $4\ell + 2$, of its mapping tori and of manifolds bounded by the latter. We also exhibit the compatibility relations satisfied by these quadratic refinements. This leads in particular to a new $\mathbb{Z}_2$-valued topological invariant for spin manifolds of dimension $4\ell + 2$ when $\ell = 0, 2$ or is odd. The motivation for this work comes from the physics of the self-dual field theory in dimension 6, and we explain the use of our results to the study of global gravitational anomaly cancellation involving the self-dual field theory.

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Contents

1 Introduction and summary

Certain categories of manifolds admit a functorial lift of their Wu class from cohomology with $\mathbb{Z}_2$ coefficients to integral cohomology, typically in terms of characteristic classes. In this paper, we show that the existence of a functorial lift of the Wu class in degree $2\ell + 2$ allows to define canonical quadratic refinements of the pairings L_M, L_{M_ϕ} and L_W, defined as follows (Section 2.3).

- Given a manifold M of dimension $4\ell+2$, L_M is the $\mathbb{Q}/\mathbb{Z}$-valued pairing on $H^{2\ell+1}(M,\mathbb{Z})\otimes \mathbb{Z}_2$ given by half the cup product pairing modulo 1.
- Given a mapping torus M_ϕ associated to a diffeomorphism ϕ of M, L_{M_ϕ} is the linking pairing on the torsion cohomology $H^{2\ell+2}_{\rm tors}(M_\phi,\mathbb{Z})$.
- Given a manifold W bounded by M_ϕ, let $F = {\rm Im}\big(H^{2\ell+2}_{\rm free}(W, M_\phi,\mathbb{Z}) \to H^{2\ell+2}_{\rm free}(W,\mathbb{Z})\big)$ and F^* the dual space with respect to the cup product pairing. L_W is a certain $\mathbb{Q}/\mathbb{Z}$-valued pairing on F^*/F (see Section 2.3 for the details).

These quadratic refinements also satisfy interesting compatibility conditions that we spell out.

The main example of a category of manifolds admitting a functorial lift is the category of spin manifolds, for which the lift is expressed in terms of the Pontryagin classes. For spin manifolds of dimensions 2, 6, 10 and $8\ell + 6$, where a certain obstruction to our constructions vanishes, the Arf invariant of the canonical quadratic refinement of L_M provides a $\mathbb{Z}_2$-valued topological invariant. The Arf invariant of the canonical quadratic refinement of L_{M_ϕ} provides a generalization of the Rohlin invariant mod 8 to mapping tori of spin manifolds of dimension $8\ell + 6$. Another example of a category of manifolds admitting a functorial lift of the Wu class, related to the physics of the M5-brane, is presented in Section 3.1.

The motivation for this work comes from physics, more precisely from the computation of the global gravitational anomaly of the self-dual field theory [1, 2]. The mathematical context for this problem is presented in Section 7, where we also explain the use of the results derived here. The introductions of [1, 2] provide a short overview of the physical context.

Let us summarize our results in more details. From the defining property of the Wu class, an integral lift ν of the Wu class of degree $2\ell + 2$ on a manifold of dimension $4\ell + 4$ with boundary

satisfies the equation

$$\langle z \cup z, [W, \partial W] \rangle = \langle z \cup \nu, [W, \partial W] \rangle \mod 2 , \tag{1.1}$$

where z lies in $H^{2\ell+2}(W, \partial W, \mathbb{Z})$, $[W, \partial W]$ denotes the fundamental relative homology class of W and $\langle \bullet, \bullet \rangle$ is the natural pairing between homology and cohomology. Adding twice an integral class to an integral lift yields another integral lift. If ν has a functorial expression, for instance in terms of characteristic classes, then its restriction to the boundary ∂W is necessarily given by twice an integral class μ, as the Wu class of degree $2\ell + 2$ of a manifold of dimension $4\ell + 3$ vanishes. Extending μ to W allows to define a integral lift $\lambda := \nu - 2\mu$ that is trivial on the boundary. [2] If an appropriate differential structure is chosen (for instance a Riemannian metric in the case of spin manifolds), the integral lift can be supplemented by a form lift $\underline{\lambda}$ vanishing on ∂W, whose de Rahm cohomology class coincides with the de Rahm cohomology class of λ. The form lift actually allows to define a class in the relative de Rahm cohomology of W with respect to ∂W. We call a relative lift the pair formed by an integral lift trivial on ∂W and a compatible form lift vanishing on ∂W (Definition 3.3).

We now sketch how relative lifts of the Wu class can be used to define quadratic refinements (Definition 2.2).

Manifolds of dimension $4\ell + 2$ Consider a manifold W whose boundary is of the form $M \times S^1$ for a $4\ell + 2$-dimensional manifold M. Let L_M be half the cup product pairing on $H^{2\ell+1}(M, \mathbb{Z})$ modulo 1. L_M can be seen as a $\mathbb{Q}/\mathbb{Z}$-valued pairing on $H^{2\ell+1}(M, \mathbb{Z}) \otimes \mathbb{Z}_2$. A relative lift $(\lambda, \underline{\lambda}) \in H^{2\ell+2}(W, M \times S^1, \mathbb{Z}) \times \Omega^{2\ell+2}(W)$ of the Wu class on W can be used to construct a function

$$Q(x) = \frac{1}{2} \int_W \underline{z} \wedge (\underline{z} - \underline{\lambda}) \mod 1 . \tag{1.2}$$

where $x \in H^{2\ell+1}(M, \mathbb{Z})$, $\underline{z}$ is a form representative of an integral class z extending $x \cup t$ to W, and t is the generator of $H^1(S^1, \mathbb{Z})$. Q is a quadratic refinement of L_M (Proposition 5.4). It is independent of the choice of bounded manifold W as well as of the various choices involved in extending cohomological data from $M \times S^1$ to W (Proposition 5.2). However Q does depend on the choice of class μ used to define λ and is not canonical. The ambiguity is parameterized by the 2-torsion group $H^{2\ell+2}_{2-\mathrm{tors}}(M, \mathbb{Z})$. [3]

[2] In this paper, we will always assume that such extensions are possible. We show in Appendix A that in the case of spin manifolds with $\ell = 0, 2$ or ℓ odd, all the relevant obstructions vanish. This includes the cases of spin manifolds with boundary of dimension 4, 8 and 12, which are the relevant ones for physical applications.

[3] The idea of considering the function (1.2) on $H^{2\ell+1}(M, \mathbb{Z})$ associated to a lift of the Wu class goes back to Witten [3]. His function however does not use a relative lift, and is not a quadratic refinement in the sense of

As the cup product pairing vanishes identically on the 2-torsion subgroup $H^{2\ell+1}_{2-\mathrm{tors}}(M,\mathbb{Z}) \subset H^{2\ell+1}(M,\mathbb{Z}) \otimes \mathbb{Z}_2$, quadratic refinements of L_M restrict to characters of $H^{2\ell+1}_{2-\mathrm{tors}}(M,\mathbb{Z})$ and there is a preferred quadratic refinement Q^c that restricts to the trivial character. Q^c is the canonical quadratic refinement of L_M that we associate to M. It factors through a quadratic refinement of the cup product pairing on the free quotient $H^{2\ell+1}_{\mathrm{free}}(M,\mathbb{Z}) \otimes \mathbb{Z}_2$.

There is a preferred relative lift $(\lambda^c, \underline{\lambda}^c)$ of the Wu class on W (modulo twice a relative class), related to Q^c by (1.2). The construction of $(\lambda^c, \underline{\lambda}^c)$ can in fact be generalized to the case when ∂W is an arbitrary mapping torus of M (Section 4.3).

The Arf invariant of Q^c provides a topological $\mathbb{Z}_2$-valued invariant of M. This construction is of course strongly reminiscent of the Kervaire invariant [4] and its generalizations [5, 6]. Indeed, the generalized Kervaire invariant is the Arf invariant of a quadratic refinement of the cup product pairing on $H^{2\ell+1}(M,\mathbb{Z}_2)$ constructed using the Pontryagin-Thom construction (see [7] for a review). As we point out in Section 5.3, results of Lee, Miller and Weintraub [8] show that the two invariants coincide when M is spin of dimension 2 or 10 and $H^5_{\mathrm{tors}}(M,\mathbb{Z}) = 0$. It would be interesting to find out if the two invariants have a closer relation.

Mapping tori of dimension $4\ell + 3$ Recall that the Rohlin invariant of a spin manifold E of dimension $8\ell + 3$ is defined as the signature modulo 16 of a manifold W bounded by E. This invariant is well-defined, because the signature is a cobordism invariant, and is necessarily a multiple of 16 for closed spin manifolds of dimension $8\ell + 4$. A consequence of the results of Brumfiel and Morgan in [9] is that the Rohlin invariant modulo 8 is related to the Arf invariant of a certain quadratic refinement of the linking pairing on the torsion cohomology $H^{2\ell+2}_{\mathrm{tors}}(E,\mathbb{Z})$.

The results of [9] are however more general (Theorem 2.9): given any compact orientable manifold E of dimension $4\ell + 3$, they allow to express the signature modulo 8 of a bounded manifold W in terms of a quadratic refinement of the linking pairing on E and an integral lift of the Wu class on W.

When E is a mapping torus in our category, the existence of a preferred relative lift $(\lambda^c, \underline{\lambda}^c)$ on W allows to define a canonical quadratic refinement $\mathcal{Q}^c$ of the linking pairing, whose Arf invariant is a generalization of the Rohlin invariant modulo 8 (see equation (4.7)). Theorem 2.9 shows that the Arf invariant of $\mathcal{Q}^c$ has a simple expression in terms of a manifold W bounded by E, just like the Rohlin invariant modulo 8:

$$A(\mathcal{Q}^c) = \frac{1}{8}\left(\int_W \underline{\lambda}^c \wedge \underline{\lambda}^c - \sigma_W\right) \mod 1 \,, \tag{1.3}$$

Definition 2.2, see the discussion in Section 7.2.

where σ_W is the signature of the cup product pairing on $H^{2\ell+2}(W, E, \mathbb{Z})$.

Compatibility The crucial result for applications to physics is the following. A quadratic refinement Q of the pairing L_M associated to a $4\ell + 2$-dimensional manifold M induces a quadratic refinement $\mathcal{Q}_Q$ of the linking pairing of any of its mapping tori (Definition 2.6). We show that if $Q = Q^c$, the induced quadratic refinement $\mathcal{Q}_{Q^c}$ coincides $\mathcal{Q}^c$ (Theorem 5.7). In particular, the Arf invariant of $\mathcal{Q}_{Q^c}$ is given by (1.3) (Theorem 4.11).

Applications In [1, 2], we defined certain line bundles with connections over the space of metrics modulo diffeomorphism $\mathcal{M}/\mathcal{D}$ of a manifold M of dimension $4\ell + 2$. These bundles are pull-backs of theta bundles from the space of complex structures on the intermediate Jacobian, endowed with connections analogous to the Bismut-Freed connections existing on determinant bundles of Dirac operators (see Section 7 for more details). In particular, they depend on a quadratic refinement Q of L_M. It has been conjectured in [2] that the holonomy of the connection along a cycle c in $\mathcal{M}/\mathcal{D}$ is computed in terms of the associated mapping torus M_c of M by $\exp \frac{2\pi i}{8}\big(\eta_0 + A(\mathcal{Q}_Q)\big)$, where η_0 is a certain limit of the eta invariant of the signature operator on M_c, and $A(\mathcal{Q}_Q)$ is the Arf invariant of the quadratic refinement $\mathcal{Q}_Q$ of the linking pairing of M_c induced by Q. After using the Atiyah-Patodi-Singer theorem to express the eta invariant in terms of data on a bounded manifold W, the holonomy formula reads

$$\frac{1}{2\pi i} \ln \mathrm{hol}(c) = \frac{1}{8} \int_W \big(\underline{\lambda}_Q \wedge \underline{\lambda}_Q - L\big) \,, \tag{1.4}$$

where L is the Hirzebruch L-genus. In the physical context, such holonomy formulas are crucial to check the consistency of low energy effective field theories obtained from theories of quantum gravity, such as string theory. Our results show that when we pick the quadratic refinement Q on M to be Q^c, the form $\underline{\lambda}_Q$ appearing in (1.4) is precisely $\underline{\lambda}^c$. As the physically relevant choice of quadratic refinement should be the canonical one, our result make the holonomy formula (1.4) concrete. We will explore the physical consequences of this result in other publications.

Relation to the work of Hopkins and Singer Let us mention that many of the structures we consider in this paper have been described in a more abstract and arguably more elegant formalism by Hopkins and Singer in [10]. However we have not managed to make this formalism explicit enough for our purpose, neither have we been able to recast our results in their language. We note some important differences between the two works:

- In [10], a lift of the Wu class to a differential cocycle is used. As it was remarked there, there is no obvious functorial way of constructing such a lift. We only require the existence of a integral lift and a compatible form lift of the Wu class, which can be chosen in a functorial way.
- Our definition of quadratic refinements follows Brumfiel and Morgan [9] and differs from the one used in [10] and commonly found in the physics literature. In particular, Hopkins-Singer quadratic refinements admit continuous deformations, while Brumfiel-Morgan quadratic refinements are discrete objects. This rigidity is in fact crucial for our construction of a topological invariant. When the pairing to be refined is the intersection product on the middle degree cohomology of a manifold of degree $2\ell + 2$, the Hopkins-Singer quadratic refinements are in fact logarithms of semi-characters [11], associated to holomorphic line bundles on the intermediate Jacobian, while the Brumfiel-Morgan quadratic refinement are associated to symmetric holomorphic line bundles. See also our discussion in Section 7.2 of the related objects defined by Witten in [3].
- We construct and use *relative* lifts of the Wu class on manifolds of dimension $4\ell + 4$ with boundary. This fact is instrumental in the construction of quadratic refinements satisfying the definition of Brumfiel-Morgan. No similar condition appears in the main theorem of [10], although it is required in the construction of Section 5.3 of [10] that the lift restricts at most to a torsion class on the boundary.

Organization of the paper Section 2 presents the mathematical background and the definitions. The definition of quadratic refinements and some of their basic properties is found in Section 2.2. In Section 2.3, we show how quadratic refinements can be associated to manifolds of dimension $4\ell + 2$, $4\ell + 3$ and $4\ell + 4$, and define certain compatibility conditions between them. We also review the main result of Brumfiel and Morgan in [9] (Theorem 2.9).

We define integral, form and differential lifts of the Wu class in Section 3.1. The notion of a functorial lift is defined in Section 3.2 and the construction of the relative lift is presented in Section 3.3.

In Section 4, we consider mapping tori of manifolds of dimension $4\ell + 2$. We show how the existence of a relative lift of the Wu class on manifolds bounded by the mapping torus allows to define a quadratic refinement of its linking pairing in Section 4.2. We construct a canonical relative lift in Section 4.3 and study the associated canonical quadratic refinement in Section 4.4. In Section 4.5, we express its Arf invariant in terms of the relative lift and the signature of

the bounded manifold W, using Theorem 2.9 from Brumfiel and Morgan.

Section 5 is devoted to the construction of the canonical quadratic refinement of L_M for a manifold M of dimension $4\ell+2$. We present in Section 5.1 the general construction of a quadratic refinement starting from a relative lift of the Wu class on a manifold bounded by the trivial mapping torus of M. In Section 5.2, we use the canonical relative lift constructed in Section 4.3 to obtain a canonical quadratic refinement. We discuss the associated topological invariant in Section 5.3. We show in Section 5.4 that the canonical quadratic refinement associated to a mapping torus of M is induced from the quadratic refinement of L_M.

In Section 6, we perform explicit computation for $M = S^3 \times S^3$. This example shows that neither the Arf invariants of mapping tori associated to M nor the relative lift of the Wu class satisfy special evenness properties, a result of interest for physical applications.

Section 7 presents informally the mathematics underlying the computation of the global gravitational anomaly of the self-dual field theory. We explain the use of our results and comment on the relation to the physics literature.

Appendix A contains basic cobordism computations showing that the obstruction to our constructions of quadratic refinements vanishes for spin manifolds when $\ell = 0, 2$ or ℓ odd.

2 Preliminary notions

2.1 Basics and notations

A good and free reference on the material presented here is [12].

Absolute and relative cohomology Let W be a manifold with boundary ∂W. Recall that the absolute cohomology $H^\bullet(W, \mathbb{Z})$ of W is defined as the space of closed cochain[4], modulo exact ones. The relative cohomology $H^\bullet(W, \partial W, \mathbb{Z})$ is defined similarly, but restricting to cochains vanishing on ∂W. If the boundary ∂W is empty, the two cohomologies agree and what follows is still valid. We have the following long exact sequence of cohomology groups

$$... \to H^{p-1}(\partial W, \mathbb{Z}) \xrightarrow{\delta} H^p(W, \partial W, \mathbb{Z}) \xrightarrow{j} H^p(W, \mathbb{Z}) \xrightarrow{i} H^p(\partial W, \mathbb{Z}) \to ... , \tag{2.1}$$

where j arises by seeing a relative cochain as an absolute one and i is the restriction of an absolute cochain to the boundary. To construct the image by δ of a cochain on ∂W, we extend it in an arbitrary way to W (for instance by zero), and take its differential.

[4]It will not be necessary for us to pick a specific model for cohomology, but the various cocycles appearing in the text can for instance be pictured as Čech cocycles.

We will use the subscript "tors" to denote the torsion subgroup of a cohomology group, and the subscript "free" to denote the free quotient, e.g. $H^\bullet_{\text{free}}(W,\mathbb{Z}) := H^\bullet(W,\mathbb{Z})/H^\bullet_{\text{tors}}(W,\mathbb{Z})$. The free quotient coincides with the de Rahm cohomology of W.

The manifold W has a fundamental relative homology cycle of degree $d := \dim(W)$, which we write $[W, \partial W]$. It can be paired with a degree d cocycle $\hat{v}$: $\langle \hat{v}, [W, \partial W] \rangle \in \mathbb{Z}$. [5] In case $\hat{v}$ is a relative cocycle, the pairing depends only on the cohomology class v of $\hat{v}$, in which case we write $\langle v, [W, \partial W] \rangle$.

Cup product There is a cup product $\cup$ on cochains inducing products on the cohomology groups. The cup product of two absolute cochains is an absolute cochain, while the cup product of a relative cochain with any cochain is a relative cochain. The cup product induces a pairing between $H^p(W, \partial W, \mathbb{Z})$ and $H^{d-p}(W,\mathbb{Z})$: $(u, v) \to \langle u \cup v, [W, \partial W] \rangle$. This pairing induces a non-degenerate $\mathbb{Z}$-valued pairing between $H^p_{\text{free}}(W, \partial W, \mathbb{Z})$ and $H^{d-p}_{\text{free}}(W,\mathbb{Z})$. Let $\Omega^\bullet(W)$ be the space of $\mathbb{R}$-valued differential forms on W. The cup product on $H^\bullet_{\text{free}}(W,\mathbb{Z})$ coincides with the product induced from the wedge product on $\Omega^\bullet(W)$.

The cup product allows as well to define a non-degenerate $\mathbb{Q}/\mathbb{Z}$-valued "linking" pairing between $H^p_{\text{tors}}(W, \partial W, \mathbb{Z})$ and $H^{d-p+1}_{\text{tors}}(W,\mathbb{Z})$. It is defined as follow. Given cocycles $\hat{x}$ and $\hat{y}$ representing elements $x \in H^p_{\text{tors}}(W, \partial W, \mathbb{Z})$ and $y \in H^{d-p+1}_{\text{tors}}(W,\mathbb{Z})$, there exists an integer k such that $k\hat{y} = d\hat{u}$, with $\hat{u}$ a chain of degree $2\ell + 1$. Define the linking pairing as

$$L_W(x, y) = \frac{1}{k}\langle \hat{x} \cup \hat{u}, [W, \partial W] \rangle \mod 1 , \tag{2.2}$$

a well-defined pairing on the torsion cohomology.

Differential cohomology [6] Denote by $C^\bullet(W,\mathbb{K})$ the space of $\mathbb{K}$-valued cochains on W and $Z^\bullet(W,\mathbb{K})$ the corresponding cocycles. A differential cochain of degree p on W is an element of $C^p(W,\mathbb{Z}) \times C^{p-1}(W,\mathbb{R}) \times \Omega^p(W)$. The form component is called the field strength of the differential cochain. A differential is defined by

$$d(\hat{a}, \hat{h}, \underline{\omega}) := (d\hat{a}, \underline{\omega} - d\hat{h} - \hat{a}, d\underline{\omega}) , \tag{2.3}$$

where the form $\underline{\omega}$ is realized as a real cochain in the second component. Ker(d) is the space of differential cocycles. A differential cohomology class is a differential cocycle modulo differentials

[5] Our notations are as follow. A cohomology class is written v (plain), a differential cohomology class $\check{v}$ (check), a differential form $\underline{v}$ (underline) and a cochain/cocycle $\hat{v}$ (hat).

[6] See Section 2 of [13] for a pedagogical introduction to differential cocycles.

of cocycles with vanishing field strengths. We will write $\check{H}^\bullet(W)$ for differential cohomology of W. The notion of "reduced" differential cohomology will prove useful.

Definition 2.1. *Reduced differential cohomology $\check{R}^\bullet(W)$ is the additive group of pairs $\check{v} = \big(a(\check{v}), \omega(\check{v})\big) \in H^p(W, \mathbb{Z}) \times \Omega^p(W)$ satisfying $[a(\check{v})]_{\mathrm{dR}} = [\omega(\check{v})]_{\mathrm{dR}}$ in de Rahm cohomology.*

2.2 Quadratic refinements

Consider a finite abelian group G (written additively) endowed with a $\mathbb{Q}/\mathbb{Z}$-valued bilinear pairing L.

Definition 2.2. *(See for instance [9, 14]) A quadratic refinement is a function $Q : G \to \mathbb{Q}/\mathbb{Z}$ satisfying the following equations:*

$$Q(g_1 + g_2) - Q(g_1) - Q(g_2) = L(g_1, g_2) \ , \tag{2.4}$$

$$Q(ng) = n^2 Q(g) \ . \tag{2.5}$$

for $g, g_1, g_2 \in G$, $n \in \mathbb{Z}$.

Remark that this definition of quadratic refinements differs from the one usually found in the physics literature (and in [10]). Any two quadratic refinements differ by a $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$-valued character of G (Theorem 2.2 in [9]).

A subgroup $G_0 \subset G$ is called isotropic if $L|_{G_0 \times G_0}$ vanishes. From (2.4), we deduce that the restriction of a quadratic refinement to G_0 is linear. Using (2.5) as well, we can deduce that $2Q|_{G_0} = 0$. Therefore $Q|_{G_0}$ is a $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$-valued character χ of G_0. By extending χ to a $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$-valued character of G, it is always possible to construct a quadratic refinement $Q - \chi$ that vanishes on G_0.

The setup described above can be obtained from a free abelian group F of finite rank endowed with a $\mathbb{Z}$-valued symmetric pairing B. B can be seen as a map b from F to its dual F^* and we have a short exact sequence

$$0 \to F \xrightarrow{b} F^* \to G_B \to 0 \ , \tag{2.6}$$

where G_B is a finite abelian group. B induces a $\mathbb{Q}$-valued pairing $B^* := \langle b^{-1} \bullet, \bullet \rangle$ on F^* which passes to a well-defined pairing $L_B : G_B \times G_B \to \mathbb{Q}/\mathbb{Z}$. A characteristic element for B is an element $\lambda \in F$ such that $B(w, w) = B(\lambda, w) \bmod 2$ for all $w \in F$. Given a characteristic element λ, we can define a quadratic refinement

$$Q_\lambda(w + F) := \frac{1}{2}\big(B^*(w, w) - B^*(w, \lambda)\big) \mod 1 \tag{2.7}$$

of L_B. The set of modulo 2 reductions of characteristic elements is in bijection with the set of quadratic refinements (Theorem 2.4 [9][7]).

A useful quantity associated to a quadratic refinement is the Gauss sum

$$\mathrm{Gauss}(Q) = \sum_{g \in G} \exp 2\pi i Q(g) \ . \tag{2.8}$$

Suppose that L is degenerate, G_0 is the radical and $Q|_{G_0}$ is a non-trivial character. Then the Gauss sum (2.8) vanishes. Indeed, if $x_0 \in G_0$ is such that $Q(x_0) = \frac{1}{2}$, then the contributions of x and $x + x_0$ cancel in pairs in (2.8). If L is non-degenerate, it can be shown that Gauss(Q) never vanishes, and that its argument is a multiple of $2\pi/8$. The corresponding element of $A(Q) \in \frac{1}{8}\mathbb{Z}/\mathbb{Z}$ is called the Arf invariant of Q. In the case when the quadratic refinement is obtained from a characteristic element λ, a theorem of van der Blij [15] computes the Gauss sum of Q_λ in terms of λ:

$$\mathrm{Gauss}(Q_\lambda) = |G_B|^{1/2} \exp \frac{2\pi i}{8} \big(\sigma_B - B(\lambda, \lambda)\big) \ , \tag{2.9}$$

where σ_B is the signature of B.

2.3 Quadratic refinements in topology and compatibility conditions

In this section we describe constructions of quadratic refinements associated to manifolds of dimension $4\ell + 2$, $4\ell + 3$ and $4\ell + 4$, as well as natural compatibility conditions between these quadratic refinements.

Dimension $4\ell + 2$ Consider a manifold M of dimension $4\ell + 2$. Consider the abelian group $G_M = H^{2\ell+1}(M, \mathbb{Z}) \otimes \mathbb{Z}_2$. Endow it with the $\mathbb{Q}/\mathbb{Z}$-valued bilinear pairing

$$L_M(x_1, x_2) := \frac{1}{2}\langle x_1 \cup x_2, [M]\rangle \mod 1 \ . \tag{2.10}$$

The cup product on the degree $2\ell + 1$ cohomology of M is antisymmetric. Because it is valued in $\left\{0, \frac{1}{2}\right\}$, L_M is also symmetric, so it can admit quadratic refinements. The set of all the quadratic refinements of such half-integral antisymmetric pairings is easy to characterize. Recall that adding a $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$-valued character of G_M to a quadratic refinement yields another quadratic refinement. As any character of G_M is $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$-valued, the set of quadratic refinement is a torsor on the group of characters of G_M. We will define a canonical quadratic refinement of L_M in Section 5.

[7] Characteristic elements are called Wu classes in that paper.

Dimension $4\ell+3$ Given a manifold E of dimension $4\ell+3$, we can consider quadratic refinements of the linking pairing L_E on $H^{2\ell+2}_{\rm tors}(E,\mathbb{Z})$ and the associated Arf invariants. In the case of a spin manifold of dimension $8\ell+3$, for a special choice of quadratic refinement, the Arf invariant coincides with the reduction modulo 8 of the Rohlin invariant, which is defined as the signature modulo 16 of a manifold bounded by E [8]. In Section 4, we will construct a canonical quadratic refinement of L_E when E is a mapping torus.

Mapping tori Recall that the mapping torus associated to a $4\ell+2$ dimensional manifold M and a diffeomorphism ϕ of M is the quotient M_ϕ of $M\times\mathbb{R}$ by the equivalence relation $(x,s)\simeq(\phi(x),s+1)$. M_ϕ can be seen as a fiber bundle over a circle with fiber M. The cohomology of M_ϕ admits a canonical decomposition (cf. the appendix of [2])

$$H^p(M_\phi,\mathbb{Z})\simeq R^p(M_\phi)\oplus H^p_\phi(M,\mathbb{Z})\ . \tag{2.11}$$

$H^\bullet_\phi(M,\mathbb{Z})$ denotes the part of the cohomology of M left invariant by the action ϕ^* of ϕ by pull-backs. $R^p(M_\phi)$ admits a filtration

$$H^{p-1}_\phi(M,\mathbb{Z})\subset R^p(M_\phi) \tag{2.12}$$

with associated graded $H^{p-1}_\phi(M,\mathbb{Z})\oplus T^{p-1}(M_\phi)$, where $T^{p-1}(M_\phi)$ is a finite group given by the quotient of $H^{p-1}(M,\mathbb{Z})/H^{p-1}_\phi(M,\mathbb{Z})$ by the image of $1-\phi^*$.

Consider the torsion subgroup

$$\begin{aligned} H^{2\ell+2}_{\rm tors}(M_\phi,\mathbb{Z}) &\simeq R^{2\ell+2}_{\rm tors}(M_\phi)\oplus H^{2\ell+2}_{{\rm tors},\phi}(M,\mathbb{Z}) \\ &\simeq H^{2\ell+1}_{{\rm tors},\phi}(M,\mathbb{Z})\oplus T^{2\ell+1}(M_\phi)\oplus H^{2\ell+2}_{{\rm tors},\phi}(M,\mathbb{Z})\ , \end{aligned} \tag{2.13}$$

where the second decomposition is a priori not canonical. The linking pairing L_{M_ϕ} identifies the first and third summands as dual of each other. The second decomposition can be made canonical by identifying $T^{2\ell+1}(M_\phi)$ with the subgroup of $H^{2\ell+2}_{\rm tors}(M_\phi,\mathbb{Z})$ that has vanishing linking pairing with elements in $H^{2\ell+1}_{{\rm tors},\phi}(M,\mathbb{Z})\oplus H^{2\ell+2}_{{\rm tors},\phi}(M,\mathbb{Z})$. In turn, using the inclusion $H^{2\ell+2}_{\rm tors}(M_\phi,\mathbb{Z})\subset H^{2\ell+2}(M_\phi,\mathbb{Z})$, this allows to define a canonical decomposition

$$H^{2\ell+2}(M_\phi,\mathbb{Z})\simeq H^{2\ell+1}_\phi(M,\mathbb{Z})\oplus T^{2\ell+1}(M_\phi)\oplus H^{2\ell+2}_\phi(M,\mathbb{Z})\ . \tag{2.14}$$

L_{M_ϕ} induces a non-degenerate pairing L_T on $T^{2\ell+1}(M_\phi)$. Let $y,y'\in T^{2\ell+1}(M_\phi)$. Assume they lift to $x,x'\in H^{2\ell+1}(M,\mathbb{Z})$. Suppose y' has order k, so there exists $u'\in H^{2\ell+1}(M,\mathbb{Z})$ such that $kx'=(1-\phi^*)u'$. The linking pairing L_T can be computed as follows:

Proposition 2.3. *(Proposition 2.1.1 of [8])*

$$L_T(y, y') = \frac{1}{k}\big\langle x \cup u', [M]\big\rangle \mod 1 \,. \tag{2.15}$$

Compatibility Pick a quadratic refinement Q of the cup product pairing L_M on $H^{2\ell+1}(M, \mathbb{Z}) \otimes \mathbb{Z}_2$ which is ϕ-invariant, i.e. satisfying $Q(\phi^* x) = Q(x)$. Consider the following function on $T^{2\ell+1}(M_\phi)$:

$$\mathcal{Q}_T(y) = \frac{1}{2k}\big\langle x \cup u, [M]\big\rangle - Q(x) \mod 1 \,, \tag{2.16}$$

where k is the order of y and u satisfies $kx = (1 - \phi^*)u$.

Proposition 2.4. *(Proposition 2.2.1 of [8]) $\mathcal{Q}_T$ is well-defined and it is a quadratic refinement of L_T.*

We define the following natural compatibility condition.

Definition 2.5. *Given a quadratic refinement Q of the pairing L_M, a quadratic refinement $\mathcal{Q}$ of the linking pairing of a mapping torus M_ϕ is said to be compatible with Q if*

- *$\mathcal{Q} = \mathcal{Q}_T$ on $T^{2\ell+1}(M_\phi)$.*
- *$\mathcal{Q} = Q$ on $H^{2\ell+1}_{2-\mathrm{tors},\phi}(M, \mathbb{Z})$, where $H^{2\ell+1}_{2-\mathrm{tors},\phi}(M, \mathbb{Z})$ is seen as a subgroup of $H^{2\ell+2}_{\mathrm{tors}}(M_\phi, \mathbb{Z})$ on the left hand side, and of $H^{2\ell+1}(M, \mathbb{Z}) \otimes \mathbb{Z}_2$ on the right hand side.*

As $H^{2\ell+2}_{2-\mathrm{tors},\phi}(M, \mathbb{Z}) \subset H^{2\ell+2}_{2-\mathrm{tors}}(M_\phi, \mathbb{Z})$ is isotropic, one can always twist a compatible quadratic refinement $\mathcal{Q}$ with a $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$-valued character of $H^{2\ell+2}_{\mathrm{tors}}(M_\phi, \mathbb{Z})$ to make it vanish on $H^{2\ell+2}_{2-\mathrm{tors},\phi}(M, \mathbb{Z})$. This allows us to define a unique quadratic refinement of L_{M_ϕ} from Q.

Definition 2.6. *Let Q be a ϕ-invariant quadratic refinement of L_M and M_ϕ a mapping torus of M. The induced quadratic refinement $\mathcal{Q}_Q$ is the unique quadratic refinement of L_{M_ϕ} vanishing on $H^{2\ell+2}_{2-\mathrm{tors},\phi}(M, \mathbb{Z}) \subset H^{2\ell+2}_{\mathrm{tors}}(M_\phi, \mathbb{Z})$ and compatible with Q.*

Dimension $4\ell + 4$ with a boundary Let W be a manifold of dimension $4\ell + 4$ with a boundary ∂W. Let F be the image of $H^{2\ell+2}_{\mathrm{free}}(W, \partial W, \mathbb{Z})$ in $H^{2\ell+2}_{\mathrm{free}}(W, \mathbb{Z})$. The cup product pairing on $H^{2\ell+2}_{\mathrm{free}}(W, \partial W, \mathbb{Z})$ induces an integral pairing B on F. If $z_1, z_2 \in F$,

$$B(z_1, z_2) = \big\langle z_1 \cup z_2, [W, \partial W]\big\rangle \,. \tag{2.17}$$

Therefore we find ourselves in the situation described in Section 2.2. We can see the pairing B as a map b from F to its dual F^*, which is nothing but the kernel of the restriction map from

$H^{2\ell+2}_{\text{free}}(W,\mathbb{Z})$ to $H^{2\ell+2}_{\text{free}}(\partial W,\mathbb{Z})$ (see [9], Section 3). Now b^{-1} can be seen as a map from F^* into $F \otimes \mathbb{Q}$, so we can extend B to a not necessarily integral pairing B^* on F^*. The latter induces a pairing L_W on F^*/F after reduction modulo 1. Given $z_1, z_2 \in F^*$, we have

$$L_W(z_1 + F, z_2 + F) := B^*(z_1, z_2) = \big\langle z_1 \cup b^{-1} z_2, [W, \partial W] \big\rangle \mod 1 . \tag{2.18}$$

We now get an explicit expression for B^* and L_W.

Proposition 2.7. *([9], Lemma 3.4)*

$$L_W(z_1 + F, z_2 + F) = -L_{\partial W}(y_1, y_2) , \tag{2.19}$$

where $L_{\partial W}$ is the linking pairing on $H^{2\ell+2}_{\text{tors}}(\partial W, \mathcal{Z})$ and y_i the restriction of z_i to ∂W, $i = 1, 2$.

Proof. For $i = 1, 2$, let $\hat{z}_i$ be a representing cocycle for z_i and $\hat{y}_i$ be the restrictions of $\hat{z}_i$ to ∂W. As $z_i \in F^*$, $\hat{y}_i$ is torsion. Let k the smallest positive integer such that $k\hat{y}_2 = 0$. Let $\hat{u}_\partial$ be a cochain on ∂W such that $d\hat{u}_\partial = k\hat{y}_2$. Let $\hat{u}$ be an extension of $\hat{u}_\partial$ to W. Then

$$\frac{1}{k}(k\hat{z}_2 - d\hat{u}) \tag{2.20}$$

is a $\mathbb{Q}$-valued relative cocycle on W, and is a representative for $b^{-1}z_2$. We can therefore rewrite:

$$\begin{aligned} B^*(z_1, z_2) &= \left\langle \hat{z}_1 \cup \hat{z}_2 - \frac{1}{k} d(\hat{z}_1 \cup \hat{u}), [W, \partial W] \right\rangle \\ &= \big\langle \hat{z}_1 \cup \hat{z}_2, [W, \partial W] \big\rangle - \frac{1}{k} \big\langle \hat{y}_1 \cup \hat{u}_\partial, [\partial W] \big\rangle \\ &= \big\langle \hat{z}_1 \cup \hat{z}_2, [W, \partial W] \big\rangle - L_{\partial W}(y_1, y_2) \end{aligned} \tag{2.21}$$

from the definition of the linking pairing (2.2). But the first term is clearly an integer, so we obtain (2.19). ☐

Let us now pick a characteristic element λ for B, i.e. an element $\lambda \in F$ such that

$$\big\langle z \cup z, [W, \partial W] \big\rangle = \big\langle z \cup \lambda, [W, \partial W] \big\rangle \mod 1 \tag{2.22}$$

for all $z \in F$. According to the discussion in Section 2.2, we have an associated quadratic refinement of L_W on F^*/F given by

$$\mathsf{Q}_\lambda(z + F) = \frac{1}{2} \big\langle z \cup (b^{-1}z - \lambda), [W, \partial W] \big\rangle \mod 1 , \tag{2.23}$$

for $z \in F^*$. As was explained in Section 2.2, the Arf invariant of q_λ can be expressed in terms of λ as

$$A(\mathsf{Q}_\lambda) = \frac{1}{8} \left(\sigma_W - \big\langle \lambda \cup \lambda, [W, \partial W] \big\rangle \right) \mod 1 , \tag{2.24}$$

where σ_W is the signature of the $4\ell + 4$ manifold W, i.e. the signature of the bilinear form B.

Compatibility We now turn to the relations between $A(\mathsf{Q}_\lambda)$ and the Arf invariants of quadratic refinements associated with ∂W. The proofs of the claims below can be found in Section 4 of [9]. Let $\mathcal{Q}$ be a quadratic refinements of $L_{\partial W}$.

Definition 2.8. *A characteristic element λ for B is said to be compatible with $\mathcal{Q}$ if for all $z \in H^{2\ell+2}(W,\mathbb{Z})$ such that $z|_{\partial W} \in H^{2\ell+2}_{\rm tors}(\partial W,\mathbb{Z})$, we have*

$$\mathcal{Q}(z|_{\partial W}) = -\mathsf{Q}_\lambda(z) \ . \tag{2.25}$$

The image of $H^{2\ell+2}_{\rm tors}(W,\mathbb{Z})$ in $H^{2\ell+2}_{\rm tors}(\partial W,\mathbb{Z})$ is isotropic, which implies that $\mathcal{Q}$ is linear and $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$-valued on this image. Hence there exists an element $b \in H^{2\ell+2}_{\rm tors}(\partial W,\mathbb{Z})$ such that

$$\mathcal{Q}(z|_{\partial W}) = L_{\partial W}(z|_{\partial W}, b) \tag{2.26}$$

for all $z \in H^{2\ell+2}_{\rm tors}(W,\mathbb{Z})$. Brumfiel and Morgan proved the following theorem:

Theorem 2.9. *([9], Theorem 4.3) If λ is a characteristic element for B compatible with $\mathcal{Q}$, then $A(\mathsf{Q}_\lambda) = \mathcal{Q}(b) - A(\mathcal{Q})$. In other words, using* (2.24)*,*

$$A(\mathcal{Q}) - \mathcal{Q}(b) = \frac{1}{8}\left(\langle \lambda \cup \lambda, [W,\partial W]\rangle - \sigma_W\right) \quad \text{mod } 1 \ . \tag{2.27}$$

3 Wu classes, functorial lifts and relative lifts

3.1 Wu classes and differential lifts

The Wu class of a manifold W is an element $\nu_2 \in H^\bullet(W,\mathbb{Z}_2)$ whose degree k component satisfies $\nu_2^{(k)} \cup x = Sq^k(x)$, where Sq is the Steenrod square operation on mod 2 cohomology. The components $\nu_2^{(k)}$ for $k > d/2$ all vanish. If the dimension d of W is even, $\nu_2^{(d/2)}$ satisfies $x \cup \nu_2^{(d/2)} = x \cup x$ for $x \in H^{d/2}(W,\mathbb{Z}_2)$. From now on, the Wu class of a manifold will always be understood as the component $\nu_2^{(d/2)}$.

We have the long exact sequence

$$\ldots \to H^{d/2}(W,\mathbb{Z}) \xrightarrow{\cdot 2} H^{d/2}(W,\mathbb{Z}) \xrightarrow{\text{mod } 2} H^{d/2}(W,\mathbb{Z}_2) \xrightarrow{\beta} H^{d/2+1}(W,\mathbb{Z}) \xrightarrow{\cdot 2} \ldots \tag{3.1}$$

where β is the Bockstein map. This sequence implies the existence of a filtration

$$H^{d/2}_{\rm tors}(W,\mathbb{Z}) \otimes \mathbb{Z}_2 \subset H^{d/2}(W,\mathbb{Z}) \otimes \mathbb{Z}_2 \subset H^{d/2}(W,\mathbb{Z}_2) \ , \tag{3.2}$$

so we have a non-canonical decomposition

$$H^{d/2}(W,\mathbb{Z}_2) = H^{d/2}_{\rm tors}(W,\mathbb{Z}) \otimes \mathbb{Z}_2 \oplus H^{d/2}(W,\mathbb{Z})_{\rm free} \otimes \mathbb{Z}_2 \oplus H^{d/2+1}_{\rm tors}(W,\mathbb{Z}) \otimes \mathbb{Z}_2 \ . \tag{3.3}$$

The cup product on $H^{2\ell+2}(W,\mathbb{Z}_2)$ induces a non-degenerate pairing. The restriction of the cup product pairing on $H^{d/2}_{\rm free}(W,\mathbb{Z})\otimes\mathbb{Z}_2$ is non-degenerate, while its restriction on $H^{d/2}_{\rm tors}(W,\mathbb{Z})\otimes\mathbb{Z}_2$ and $H^{d/2+1}_{\rm tors}(W,\mathbb{Z})\otimes\mathbb{Z}_2$ is zero. It identifies $H^{d/2+1}_{\rm tors}(W,\mathbb{Z})\otimes\mathbb{Z}_2$ as the dual of $H^{d/2}_{\rm tors}(W,\mathbb{Z})\otimes\mathbb{Z}_2$. The cup products on $H^{d/2}(W,\mathbb{Z})$ and $H^{d/2}(W,\mathbb{Z}_2)$ are compatible, in the sense that the reduction modulo 2 of the first gives the restriction of the second on the first two summands of (3.3).

Definition 3.1. *An integral lift ν of the Wu class ν_2 is an element $\nu \in H^{2\ell+2}(W,\mathbb{Z})$ such that $\nu = \nu_2$ modulo 2.*

A form lift of the Wu class is a closed form $\underline{\nu} \in \Omega^{2\ell+2}(W)$ whose de Rahm cohomology class coincides with the de Rahm cohomology class of an integral lift.

A differential lift $\check{\nu}$ of the Wu class is an element of $\check{R}^{d/2}(W)$ such that $a(\check{\nu})$ is an integral lift.

Remarks Clearly, such lifts are possible only if the Wu class has no component in the third summand of (3.3) or equivalently if $\beta(\nu_2) = 0$.

From the definition of reduced differential cohomology classes, we deduce that the field strength of a differential lift is a form lift.

Pick an integral lift ν and a class $z \in H^{2\ell+2}(W,\mathbb{Z})$. We have

$$\langle z \cup z, [W]\rangle = \langle z \cup \nu, [W]\rangle \mod 2 \ . \tag{3.4}$$

Pick a form lift $\underline{\nu}$ and a closed form $\underline{z} \in \Omega^{2\ell+2}(W)$. We have

$$\int_W \underline{z} \wedge \underline{z} = \int_W \underline{z} \wedge \underline{\nu} \mod 2 \ . \tag{3.5}$$

3.2 Functorial lifts

Definition 3.2. *A category of manifolds admits a functorial lift of the Wu class if there is a functorial way of associating a differential lift of the degree $2\ell + 2$ Wu class to manifolds of dimension $4\ell + 3$ and to manifolds of dimension $4\ell + 4$ with boundaries.*

A "functorial way" means here that if $\check{\nu}_X$ is the differential lift associated to a manifold X and if there is a morphism $\phi : A \to B$, then $\check{\nu}_A = \phi^*(\check{\nu}_B)$. In particular, the differential lifts are compatible under the operation of restriction to the boundary of a $4\ell+4$ dimensional manifold. The typical case is when a category of manifolds admits an expression for an integral lift of the Wu class to integral cohomology in terms of characteristic classes. By considering a refined

category where the manifolds are equipped with an appropriate differential structure, one can arrange so that the characteristic classes admit canonical form representatives modulo torsion. Both combine into a differential lift of the Wu class which is functorial.

Example 3.1. Spin manifolds admit a lift ν of the Wu class in integral cohomology in terms of Pontryagin classes (see the appendix E of [10]). For instance in degree 4, $\nu = -p_1/2$, where p_1 is the first Pontryagin class. A Riemannian metric allows to construct canonical form representatives for the Pontryagin classes using its curvature. Riemannian spin manifolds therefore form a category admitting functorial lifts of the Wu classes in degree $2\ell + 2$ for each ℓ. Obviously, the functorial lifts vanish in degrees different from 0 modulo 4.

Example 3.2. The M5-brane provides another category of manifolds admitting a functorial lift of their Wu class. These are orientable manifolds W endowed with a real orientable vector bundle N. We require that N is of dimension 5, that its Euler class vanishes, that its second Stiefel-Whitney class satisfies $w_2(N) = w_2(W)$, and that it is endowed with a choice of Thom class. Together with morphisms preserving all the structures we just described, the set of such manifolds forms a category. The Thom class determines a global angular form g on 4-sphere bundle associated to N. Denote the push-forward map associated to the 4-sphere bundle by π_*. As shown in Section 5 of [16], $\pi_*(g \cup g) - p_1/2$ is an integral lift of the Wu class of degree 4. By considering manifolds equipped with a Riemannian metric on the 4-sphere bundle associated to N, we obtain a functorial lift of the Wu class of degree 4. This case will be studied further in the context of the global gravitational anomaly of the M5-brane in future work.

3.3 Relative lifts

Definition 3.3. *A relative lift of the Wu class of degree $2\ell + 2$ on a manifold W of dimension $4\ell + 4$ with boundary is a differential lift $\check{\lambda} = (\lambda, \underline{\lambda})$ such that λ is trivial when restricted to ∂W, and $\underline{\lambda}$ vanishes on ∂W.*

Remark 3.4. $\underline{\lambda} := \omega(\check{\lambda})$ is a form vanishing on the boundary with integral periods. It defines an element of the relative de Rahm cohomology $H^{2\ell+2}_{\rm free}(W, \partial W, \mathbb{Z})$. λ does not quite define an element of $H^{2\ell+2}(W, \partial W, \mathbb{Z})$, but only an element in the image of the map from $H^{2\ell+2}(W, \partial W, \mathbb{Z})$ to $H^{2\ell+2}(W, \mathbb{Z})$, which can have a non-trivial kernel (see the exact sequence (2.1)).

We now show how functorial lifts allow to construct relative lifts. Consider a category of manifolds admitting a functorial lift $\check{\nu}$ of the Wu class ν_2 of degree $2\ell + 2$. Consider a manifold E of dimension $4\ell + 3$ in this category. On E, ν_2 always vanishes, so $\check{\nu}$ can be expressed as

$\check{\nu} = 2\check{\mu}$, for $\check{\mu} \in \check{R}^{2\ell+2}_{\mathbb{Z}}(E)$. Now consider a manifold W of dimension $4\ell + 4$ bounded by E and on which $\check{\mu}$ extends. (See below for a discussion of the possible obstructions.) We still write $\check{\mu}$ for the reduced differential cohomology class extending $\check{\mu}$ on W. We can define the following relative lift of the Wu class:

$$\check{\lambda} := \check{\nu} - 2\check{\mu} \tag{3.6}$$

Remark that there are two choices required to define $\check{\lambda}$. First, $a(\check{\mu})$ is only defined up to the addition of a 2-torsion class on E. Second, we have to make a choice of extension of $\check{\mu}$ on W.

In the construction above, there are two possible obstructions. It may not be possible to find a manifold W bounded by E, and even if W exists, it may not be possible to extend $\check{\mu}$. These obstructions have to be studied case by case. In the case of spin manifolds, the first obstruction is described by the spin bordism group $\Omega^{\rm spin}_{4\ell+3}(\mathrm{pt})$, while the second one is given by $\Omega^{\rm spin}_{4\ell+3}(K(\mathbb{Z}, 2\ell+2))$. In appendix A, we show that the first obstruction vanishes for $\ell = 0, 2$ and ℓ odd. We also show that the second one always vanishes. Only in these cases our construction of the relative Wu class can be carried out for all manifolds. We do not know yet how to compute the obstruction relevant to Example 3.2.

4 A canonical quadratic refinement for mapping tori

In this section, we consider a mapping torus M_ϕ of a manifold M of dimension $4\ell + 2$. We first define quadratic refinements of the linking pairing on the torsion cohomology of degree $2\ell + 2$ associated to arbitrary relative lifts of the Wu class. Then we show how to construct a canonical relative lift, yielding a canonical quadratic refinement $\mathcal{Q}^c$.

4.1 A basic construction

The following construction will appear many times in what follows. Let E be a manifold of dimension $4\ell + 3$, endowed with a cohomology class $y \in H^{2\ell+2}(E, \mathbb{Z})$ and a reduced differential cohomology class $\check{\mu}$ satisfying $2\check{\mu} = \check{\nu}$, for $\check{\nu}$ the functorial lift of the Wu class.

Let W be a $4\ell+4$ manifold with boundary, endowed with a cohomology class $z \in H^{2\ell+2}(W, \mathbb{Z})$ and with reduced differential cohomology classes $\check{\mu}_W$ and $\check{\lambda}$.

Definition 4.1. *We say $[W, z, \check{\mu}_W, \check{\lambda}]$ is a bordism trivialization of the data $[E, y, \check{\mu}]$ if*

- *W admits E as its boundary,*
- *z restricts to y on E,*

- $\check{\mu}_W$ *restricts to* $\check{\mu}$ *on* E,

- $\check{\lambda} = \check{\nu} - 2\check{\mu}_W$, *where* $\check{\nu}$ *denotes here the functorial lift of the Wu class on* W.

We will often write $\underline{\lambda}$ for $\omega(\check{\lambda})$.

4.2 Quadratic refinements from relative lifts

Let L_{M_ϕ} be the linking pairing on $H^{2\ell+2}_{\rm tors}(M_\phi, \mathbb{Z})$. Let $\check{\mu}$ be a reduced differential cohomology class satisfying $2\check{\mu} = \check{\nu}$. Pick a torsion class $y \in H^{2\ell+2}_{\rm tors}(M_\phi, \mathbb{Z})$, let $[W_\phi, z, \check{\mu}, \check{\lambda}]$ a bordism trivialization of $[M_\phi, y, \check{\mu}]$, where by we wrote $\check{\mu}$ both for the reduced differential cohomology class on M_1 and its extension to W. Define $\lambda := a(\check{\lambda})$. Recall the bilinear form B^* described in (2.21). To ease the notation, we will write z and λ as well for the elements in $H^{2\ell+2}_{\rm free}(W, \mathbb{Z})$ defined by the corresponding integral classes. Consider the following function

$$\mathcal{Q}(y) = -\frac{1}{2}\big(B^*(z,z) - B^*(z,\lambda)\big) \quad \text{mod } 1 \ . \tag{4.1}$$

Proposition 4.2. *$\mathcal{Q}$ is independent of the choice of bordism trivialization.*

Proof. Let us assume that we have two bordism trivializations $[W_1, z_1, \check{\mu}_1, \check{\lambda}_1]$ and $[W_2, z_2, \check{\mu}_2, \check{\lambda}_2]$. Denote by B^*_1 and B^*_2 the respective bilinear pairings on W_1 and W_2 and by $\mathcal{Q}_1$ and $\mathcal{Q}_2$ the quadratic refinements obtained from (4.1).

Let $\bar{W}_2$ be W_2 with its orientation reversed. We can glue W_1 and $\bar{W}_2$ along their common boundary M_ϕ into a closed $4\ell+4$-dimensional manifold W endowed with a class y and a reduced differential cohomology class $\check{\mu}$. Consider the Mayer-Vietoris sequence

$$... \to H^{2\ell+2}(W, \mathbb{Z}) \xrightarrow{r} H^{2\ell+2}(W_1, \mathbb{Z}) \oplus H^{2\ell+2}(\bar{W}_2, \mathbb{Z}) \xrightarrow{\Delta} H^{2\ell+2}(M_\phi, \mathbb{Z}) \to ... \tag{4.2}$$

where r is the restriction map. If (u_1, u_2) and (u_1', u_2') restrict to torsion elements on M^ϕ and belong to the image of r, with preimage u and u', we deduce from (2.21) that

$$B^*_1(u_1, u_1') - B^*_2(u_2, u_2') = B(u, u') \ , \tag{4.3}$$

where the minus sign comes from the orientation reversal on W_2. Therefore,

$$\mathcal{Q}_1(y) - \mathcal{Q}_2(y) = -\frac{1}{2}\big(B(z,z) - B(z,\lambda)\big) \quad \text{mod } 1 \ . \tag{4.4}$$

As $\check{\lambda} = \check{\nu} + 2\check{\mu}$, λ differs from $[a(\nu)]_{\rm dR}$ by twice an integral class, so is a lift of the Wu class of W in de Rahm cohomology. This ensures that (4.4) is equal to 0 modulo 1. ◻

Remark 4.3. It is crucial in the argument above that the same reduced character $\breve{\mu}$ on M_1 is used in the construction of λ_1 and λ_2. Indeed, if λ_1 and λ_2 are constructed from two different characters $\breve{\mu}_1$ and $\breve{\mu}_2$, with $a(\breve{\mu}_1)$ differing from $a(\breve{\mu}_2)$ by a 2-torsion class on M_1, nothing ensures that λ differs from $\omega(\breve{\nu})$ by twice an integral class and (4.4) does not necessarily vanish. Therefore, $\mathcal{Q}$ *does* depend on $\breve{\mu}$.

Proposition 4.4. *$\mathcal{Q}$ is a quadratic refinement of L_{M_ϕ}.*

Proof. Seen as a function of z, $-\mathcal{Q}$ is obviously a quadratic refinement of the pairing L_B (see (2.7) and Section 2.2). But we showed in Proposition 2.7 that given two classes z and z' on W restricting to torsion classes y and y' on $\hat{M}$, $L_B(z,z') = -L_{\hat{M}}(y,y') \bmod 1$, proving the proposition. □

4.3 The canonical relative lift

According to (2.14), the cohomology in degree $2\ell+2$ of M_ϕ admits the canonical decomposition

$$H^{2\ell+2}(M_\phi,\mathbb{Z}) \simeq H^{2\ell+1}_\phi(M,\mathbb{Z}) \oplus T^{2\ell+1}(M_\phi) \oplus H^{2\ell+2}_\phi(M,\mathbb{Z}) \ . \tag{4.5}$$

Consider first the component $T^{2\ell+1}(M_\phi)$. As the integral lift of the Wu class $a(\breve{\nu})$ is even, it has no component on the 2-torsion subgroup $T^{2\ell+1}_{2-\mathrm{tors}}(M_\phi)$. We can choose $\breve{\mu}$ so that $a(\breve{\mu})$ has no component on $T^{2\ell+1}_{2-\mathrm{tors}}(M_\phi)$. We obtain a quadratic refinement $\mathcal{Q}$ from $\breve{\mu}$ using (4.1). We now evaluate $\mathcal{Q}$ on the subspace $H^{2\ell+1}_{2-tors,\phi}(M,\mathbb{Z})$ and $H^{2\ell+2}_{2-tors,\phi}(M,\mathbb{Z})$ of $H^{2\ell+2}_{\mathrm{tors}}(M_\phi,\mathbb{Z})$. As these subspaces are isotropic, $\mathcal{Q}$ restricts to characters χ_1 and χ_2. As they are dual to each other with respect to the linking pairing, χ_1 determines a class $u_1 \in H^{2\ell+2}_{2-tors,\phi}(M,\mathbb{Z})$ and χ_2 determines a class $u_2 \in H^{2\ell+1}_{2-tors,\phi}(M,\mathbb{Z})$:

$$\chi_i(y) = L_{M_\phi}(y,u_i) \ . \tag{4.6}$$

We define $\breve{\mu}^c$ such that $2\breve{\mu}^c = \breve{\nu}$ and $a(\breve{\mu}^c) = a(\breve{\mu}) + u_1 + u_2$. As the 2-torsion component of $a(\breve{\mu})$ is completely fixed this provides a canonical choice for $\breve{\mu}$. We write $\check{\lambda}^c$ for the associated relative lift of the Wu class.

4.4 The canonical quadratic refinement

Using the canonical relative lift in (4.1), we obtain a canonical quadratic refinement $\mathcal{Q}^c$:

$$\mathcal{Q}^c(y) = -\frac{1}{2}\big(B^*(z,z) - B^*(z,\lambda^c)\big) \mod 1 \ , \tag{4.7}$$

where λ^c is a shorthand for $a(\check{\lambda}^c)$.

Proposition 4.5. *$\mathcal{Q}^c$ vanishes on $H^{2\ell+1}_{2-tors,\phi}(M,\mathbb{Z})$ and on $H^{2\ell+2}_{2-tors,\phi}(M,\mathbb{Z})$, seen as subspaces of $H^{2\ell+2}_{\rm tors}(M_\phi,\mathbb{Z})$.*

Proof. Pick a class $y \in H^{2\ell+2}_{\rm tors}(M_\phi,\mathbb{Z})$ and a bordism trivialization $[W_\phi, z, \breve{\mu}^c, \check{\lambda}^c]$ of $[M_\phi, y, \breve{\mu}^c]$. We choose it so that u_1 and u_2 extend to classes v_1 and v_2 on W. As explained at the end of Appendix A, this is always possible provided two cohomology classes can be extended, what is necessary for the existence of a bordism trivialization, and assumed throughout this paper.

From (4.1), we have

$$\mathcal{Q}^c(y) = \mathcal{Q}(y) + \frac{1}{2}B^*(y, 2v_1 + 2v_2) \tag{4.8}$$

$$= \mathcal{Q}(y) - L_{M_\phi}(y, u_1 + u_2) \ , \tag{4.9}$$

where we used (2.21). Clearly, if y belong to the subgroups $H^{2\ell+1}_{2-tors,\phi}(M,\mathbb{Z})$ or $H^{2\ell+2}_{2-tors,\phi}(M,\mathbb{Z})$, the definition of the classes u_1 and u_2 imply that $\mathcal{Q}^c(y) = 0$. ☐

4.5 A canonical quadratic refinement for manifolds bounded by mapping tori

Let M_ϕ be a $4\ell+3$ dimensional mapping torus and let $[W, z, \breve{\mu}^c, \check{\lambda}^c]$ be a bordism trivialization of $[M_\phi, y, \breve{\mu}^c]$. Write $\lambda^c := a(\check{\lambda}^c)$. Recall that we called F the image of $H^{2\ell+2}_{\rm free}(W, M_\phi,\mathbb{Z})$ in $H^{2\ell+2}_{\rm free}(W,\mathbb{Z})$, and F^* the kernel of the restriction map from $H^{2\ell+2}_{\rm free}(W,\mathbb{Z})$ to $H^{2\ell+2}_{\rm free}(M_\phi,\mathbb{Z})$. As shown in Section 2.3, λ^c allows to define a quadratic refinement of the pairing L_W on F^*/F by

$$\mathsf{Q}^c(z) = \frac{1}{2}\big(B^*(z,z) - B^*(z,\lambda^c)\big) \mod 1 \ . \tag{4.10}$$

The following lemma is obvious from the definitions.

Lemma 4.6. Q^c *is compatible with* $\mathcal{Q}^c$ *in the sense of Definition 2.8.*

Write again $\underline{\lambda}^c := \omega(\check{\lambda}^c)$. The following result is easily derived, but crucial for applications to the self-dual field theory.

Theorem 4.7.

$$A(\mathsf{Q}^c) = \frac{1}{8}\left(\int_W \underline{\lambda}^c \wedge \underline{\lambda}^c - \sigma_W\right) \mod 1 \ . \tag{4.11}$$

Proof. The element $b \in H^{2\ell+2}_{\rm tors}(\partial W,\mathbb{Z})$ defined in (2.26) vanishes, because $\mathsf{Q}^c(z)$ obviously vanishes for z a torsion class. (4.11) is then an immediate consequence of Theorem 2.9. ☐

As was explained in the introduction, $A(\mathcal{Q}^c) = -A(\mathsf{Q}^c)$ provides a generalization of the Rohlin invariant modulo 8 to mapping tori of dimension $4\ell+3$ in our category.

5 A canonical quadratic refinement in dimension $4\ell+2$

In this section, given a $4\ell+2$-dimensional manifold M and a functorial lift of the Wu class, we construct a canonical quadratic refinement of the pairing L_M on $H^{2\ell+1}(M,\mathbb{Z})\otimes\mathbb{Z}_2$.

5.1 Preliminary definition

Pick a class $x\in H^{2\ell+1}(M,\mathbb{Z})$ and construct the manifold $M_1 := M\times S^1$ endowed with $x\cup t$, t being the class generating $H^1(S^1,\mathbb{Z})$. Following Section 3.3, pick $\breve{\mu}$ satisfying $\breve{\nu}=2\breve{\mu}$ on M_1 and let $[W,z,\breve{\mu},\check{\lambda}]$ be a bordism trivialization of $[M_1,x\cup t,\breve{\mu}]$. Let $\underline{z}$ be a representative in $\Omega^{2\ell+2}(W)$ for the image of z in $H^{2\ell+2}_{\text{free}}(W,\mathbb{Z})$.

Consider the following function on $H^{2\ell+1}(M,\mathbb{Z})$:

$$Q(x)=\frac{1}{2}\int_W \underline{z}\wedge(\underline{z}-\underline{\lambda}) \mod 1\ . \tag{5.1}$$

Remark 5.1. Although there are obvious similarities with the definition in (4.1), remark that $x\cup t$ is not necessarily torsion. (4.1) would not make sense for a class z restricting to a non-torsion form on M_1.

The following proposition shows that Q is well-defined.

Proposition 5.2. *Q depends neither on the choice of bordism trivialization $[W,z,\breve{\mu},\check{\lambda}]$ nor on the choice of form lift $\underline{z}$, and therefore yields a well-defined function on $H^{2\ell+1}(M,\mathbb{Z})$.*

Proof. Assume that we have two bordism trivializations $[W_1,z_1,\breve{\mu}_1,\check{\lambda}_1]$ and $[W_2,z_2,\breve{\mu}_2,\check{\lambda}_2]$ with corresponding forms $\underline{z}_1$, $\underline{\lambda}_1$ on W_1 and $\underline{z}_2$, $\underline{\lambda}_2$ on W_2. On M_1, we have $\underline{z}_2-\underline{z}_1=d\underline{u}$. We construct a manifold $W_3\simeq M_1\times[0,1]$ endowed with the form $\underline{z}_3=\underline{z}_1+d((3s^2-2s^3)\underline{u})$, where s parameterizes $[0,1]$. $\underline{z}_3$ coincides with $\underline{z}_1$ on $M_1\times\{0\}$ and with $\underline{z}_2$ on $M_1\times\{1\}$. Using that both $\underline{z}_1$ and $\underline{z}_2$ are of the form $\underline{x}\wedge\underline{t}$, one can check that $\underline{z}_3\wedge\underline{z}_3=0$. We can now glue W_1 and W_2 to W_3 to obtain a manifold W_{132} with a form $\underline{z}_{132}$.

We also get a form $\underline{\lambda}_{132}$ by extending $\underline{\lambda}_1$ and $\underline{\lambda}_2$ by 0 on W_3. From the construction of the form $\underline{\lambda}$ in Section 3.3 and the functoriality of $\breve{\nu}$, we see that the de Rham cohomology class $[\underline{\lambda}_{132}]_{\text{dR}}$ differs from $[\omega(\breve{\nu})]_{\text{dR}}$ by twice an integral class. Therefore $\underline{\lambda}_{132}$ is a form lift of the Wu class and satisfies (3.5).

Denote by $Q_i(x)$, $i=1,2$ the value of $Q(x)$ computed from the two bordism trivializations. We have

$$Q_1(x)-Q_2(x)=\frac{1}{2}\big\langle \underline{z}_{132}\cup(\underline{z}_{132}-\underline{\lambda}_{132}),[W_{132}]\big\rangle\ , \tag{5.2}$$

where we used the fact that the contribution of the right hand side from W_3 vanishes. (5.2) vanishes modulo 1 because of the property of the form lift (3.5). $Q_1(x)$ and $Q_2(x)$ therefore coincide. $\square$

Remark 5.3. Again, just like $\mathcal{Q}$, Q *does* depend on the choice for $\check{\mu}$.

Proposition 5.4. *Q, as defined in equation* (5.1)*, is a quadratic refinement of the pairing L_M on $H^{2\ell+1}(M,\mathbb{Z}_2)\otimes\mathbb{Z}_2$.*

Proof. From the definition of L_M in (2.10), we have to check that given $u, v \in H^{2\ell+1}(M,\mathbb{Z})$,

$$Q(u+v) - Q(u) - Q(v) = \frac{1}{2}\langle u \cup v, [M]\rangle \mod 1 \ . \tag{5.3}$$

Let P be a trinion (a three-holed sphere), with boundary components C_1, C_2 and C_3. We give C_3 an orientation opposite to the orientation of C_1 and C_2, relative to the orientation of P. Let also $W_3 = M \times P$. We endow the three boundary components $M \times C_i$ of W_3 respectively with the classes $u \cup t$, $v \cup t$ and $(u+v) \cup t$. Applying again the construction of Section 3.3, we obtain a relative lift $\underline{\lambda} \in \Omega^{2\ell+2}(W_3)$.

In order to be able to write explicit formulas, it is useful to picture the trinion as a 2-simplex $\{(x,y) \subset \mathbb{R}^2 | 0 \leqslant x \leqslant y \leqslant 1\}$ whose three vertices $(0,0)$, $(0,1)$ and $(1,1)$ have been identified [17]. This allows to write very easily an extension of the boundary classes to the interior of W_3. Let $\underline{u}$ and $\underline{v}$ be form representatives of the de Rahm cohomology classes of u and v. Write

$$\underline{z} = dx \wedge \underline{u} + dy \wedge \underline{v} \ . \tag{5.4}$$

The left-hand side of (5.3) is given by

$$\frac{1}{2}\int_{W_3} \underline{z} \wedge (\underline{z} - \underline{\lambda}) \ . \tag{5.5}$$

As P is an orientable manifold, its first Stiefel-Whitney class, which coincides with its degree 1 Wu class, vanishes. As the Wu class of degree $2\ell+2$ of M vanishes for dimensional reasons, this implies from the Cartan formula (see appendix E of [10]) that the degree $2\ell+2$ Wu class of W_3 vanishes. Hence both $[\omega(\check{\nu})]_{\mathrm{dR}}$ and $[\underline{\lambda}]_{\mathrm{dR}}$ can be expressed as twice integral classes. So modulo 1, we can drop the term involving λ in (5.3).

We can then compute

$$\frac{1}{2}\int_{W_3} \underline{z} \wedge \underline{z} = -\int_P dx \wedge dy \cdot \langle u \cup v, [M]\rangle = \frac{1}{2}\langle u \cup v, [M]\rangle \mod 1 \ , \tag{5.6}$$

which proves the proposition. $\square$

5.2 The canonical quadratic refinement

The canonical relative lift $\check{\lambda}^c$ constructed in section 4.3 provides a canonical form lift $\underline{\lambda}^c = \omega(\check{\lambda}^c)$. Using the same notation as in (5.1), we define:

$$Q^c(x) = \frac{1}{2}\int_W \underline{z} \wedge (\underline{z} - \underline{\lambda}^c) \mod 1 . \tag{5.7}$$

Propositions 5.2 and the unicity of the canonical relative lift $\check{\lambda}^c$ imply that Q^c is canonically defined from the functorial lift $\check{\nu}$ of the Wu class.

Proposition 5.5. *The restriction of Q^c to $H^{2\ell+1}_{\mathrm{tors}}(M,\mathbb{Z}) \otimes \mathbb{Z}_2$ vanishes.*

Proof. Remark that if $x \in H^{2\ell+1}_{\mathrm{tors}}(M,\mathbb{Z})$, $y := x \cup t \in H^{2\ell+1}_{\mathrm{tors}}(M,\mathbb{Z}) \subset H^{2\ell+2}_{\mathrm{tors}}(M_1,\mathbb{Z})$. In this case,

$$Q^c(x) = \frac{1}{2}\int_W \underline{z} \wedge (\underline{z} - \underline{\lambda}^c) = \frac{1}{2}\big(B^*(z,z) - B^*(z,\lambda)\big) = -\mathcal{Q}^c(y) \mod 1 . \tag{5.8}$$

We deduce from Proposition 4.5 that $Q^c(x)$ vanishes on $H^{2\ell+1}_{\mathrm{tors}}(M,\mathbb{Z}) \otimes \mathbb{Z}_2$. □

5.3 A topological invariant

The functorial lift of the Wu class $\check{\nu}$ generally depends on a differential structure on the manifold M. In the case of spin manifolds (Example 3.1 in Section 3.2), this structure is a Riemannian metric on M. As quadratic refinements do not admit continuous deformation, and as the space of Riemannian metric is simply connected (it is in fact contractible [18]), Q^c is a topological invariant in this case. Any explicit expression for Q^c depends however on a choice of basis of $H^{2\ell+1}(M,\mathbb{Z}) \otimes \mathbb{Z}_2$, which cannot be made canonically. Practically, we can obtain a computable topological invariant from its Arf invariant, as the latter is invariant under changes of bases.

As we saw in Section 2.2, when a quadratic refinement is non-trivial on the radical of the pairing it refines, its Gauss sum vanishes and the Arf invariant cannot be defined. In the case treated in this section, this happens when χ is a non-trivial character (For essentially the same reason, the partition function of the self-dual field is identically zero in this case [16].) However Q^c vanishes on $H^{2\ell+1}_{\mathrm{tors}}(M,\mathbb{Z}) \otimes \mathbb{Z}_2$ and factors through a quadratic refinement of the non-degenerate cup product pairing on $H^{2\ell+1}_{\mathrm{free}}(M,\mathbb{Z}) \otimes \mathbb{Z}_2$. In consequence, its Gauss sum does not vanish and it has a well-defined Arf invariant.

As Q^c is valued in $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$, $A(Q^c)$ is valued in $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$ as well. Therefore we have built a $\mathbb{Z}_2$ valued topological invariant associated to spin manifolds of dimension $4\ell + 2$. Because of the potential obstruction to constructing the bounded manifold W, this construction can be carried out for all manifolds only when $\ell = 0, 2$ or ℓ is odd (see the Appendix).

A natural quadratic refinement Q_{B} of the cup product pairing on $H^{2\ell+1}(M, \mathbb{Z}_2)$ has been defined by Brown using the Pontryagin-Thom construction. The generalized Kervaire invariant is defined as the Arf invariant of Q_{B} [6]. In the case when M is a spin manifold of dimension $8\ell + 2$ such that $M \times S^1$ is a spin boundary and $H^{2\ell+1}(M, \mathbb{Z})$ is torsion-free, results from Lee, Miller and Weintraub ([8], Sections 5 and 6) show that Brown's quadratic refinement coincides with (5.1) with $\underline{\lambda} = 0$. It would be very interesting to know their relation in more general cases. Remark that when $H^{2\ell+1}(M, \mathbb{Z})$ has 2-torsion, the domains of the two quadratic refinements are different.

For other categories of manifolds admitting a functorial lift of the Wu class, the canonical quadratic refinement defines a local system over the space of differential structures required to define the functorial lift. If the local system is trival, for instance when the space of differential structures is simply connected, the Arf invariant of the canonical quadratic refinement provides a topological invariant.

5.4 Compatibility

Now we show that the quadratic refinement $\mathcal{Q}^c$ is compatible with Q^c, in the sense of Definition 2.5. To this end, we need a preliminary lemma.

Pick a class $y_\phi \in H^{2\ell+2}(M_\phi, \mathbb{Z})$ that pulls back to a class $x \cup t \in H^{2\ell+2}_{\mathrm{cpt}}(M \times \mathbb{R}, \mathbb{Z})$ under the quotient map $M \times \mathbb{R} \to M_\phi$, with $x \in H^{2\ell+1}(M, \mathbb{Z})$. Let $\check{\mu}^c_\phi$ be the reduced differential cohomology class constructed in Section 4.3. Let $[W_\phi, z, \check{\mu}^c_\phi, \check{\lambda}^c_\phi]$ a bordism trivialization of $[M_\phi, y_\phi, \check{\mu}^c_\phi]$. Let $\hat{z}_\phi$ be a cocycle representing the class z_ϕ, and $\hat{\lambda}^c_\phi$ a relative cocycle satisfying $[\hat{\lambda}^c_\phi]_{\mathrm{dR}} = [\omega(\check{\lambda}^c_\phi)]_{\mathrm{dR}}$ as relative de Rahm cohomology classes.

Repeat the construction of the previous paragraph with the identity diffeomorphism $\phi = 1$.

Lemma 5.6.

$$\frac{1}{2}\big\langle \hat{z}_\phi \cup (\hat{z}_\phi - \hat{\lambda}^c_\phi), [W_\phi, M_\phi]\big\rangle = \frac{1}{2}\big\langle \hat{z}_1 \cup (\hat{z}_1 - \hat{\lambda}^c_1), [W_1, M_1]\big\rangle \mod 1\ . \tag{5.9}$$

In other words, the left-hand side is independent from ϕ modulo 1.

Proof. The main idea of the proof (loosely inspired by the proof of proposition 5.4.1 in [8]), is to choose the cocycles $\hat{z}_\phi$ and $\hat{z}_1$ so that their support on the boundary is a tubular neighborhood of a fiber of M_ϕ and M_1, and to compute both sides in a neighborhood of the support of $\hat{z}_\phi$ and $\hat{z}_1$.

Let us first remark that the left-hand side of (5.9) cannot depend on the choice of cocycle representative $\hat{z}_\phi$, by an argument completely analogous to the proof of Proposition 5.2. Let

$\hat{t}$ be a "bump cocycle" representing t and supported on an interval $I \subset S^1$. If $\hat{x}$ is a cocycle representative for x, then $\hat{z}_\phi$ can be chosen such that $\hat{z}_\phi|_{M_\phi}$ is represented by $\hat{x} \cup \hat{t}$. $\hat{z}_1$ can be taken as well to be such that $\hat{z}_1|_{M_1}$ is equal to $\hat{x} \cup \hat{t}$. Pick also cocycle representatives $\hat{\mu}^c_1$ and $\hat{\mu}^c_\phi$ of $a(\check{\mu}^c_1)$ and $a(\check{\mu}^c_\phi)$.

Let T_1 and T_ϕ be tubular neighborhoods of the supports of $\hat{z}_1$ and $\hat{z}_\phi$. $T_1 \cap M_1$ is homeomorphic to $T_\phi \cap M_\phi$ and we can assume that this homeomorphism maps $\hat{z}_\phi|_{T_\phi \cap M_\phi}$ to $\hat{z}_1|_{T_1 \cap M_1}$, as well as $\hat{\mu}^c_\phi$ to $\hat{\mu}^c_1$. Inverting the orientation of T_1, we can glue T_1 and T_ϕ along their boundaries $T_1 \cap M_1$ and $T_\phi \cap M_\phi$ to obtain a (non-compact) manifold T. $\hat{z}_\phi$ and $\hat{z}_1$ combine to form a compactly supported cocycle $\hat{z}$ in T. Similarly, $\hat{\mu}^c_1$ and $\hat{\mu}^c_\phi$ combine into a non-compactly supported cocycle $\hat{\mu}^c$. Writing $\check{\nu}$ for the functorial lift of the Wu class on T, we can define a (non-compactly supported) cocycle $\hat{\lambda}^c = a(\check{\nu}) - 2\hat{\mu}^c$. $\hat{\lambda}^c$ is a cocycle representative of an integral lift of the Wu class.

The difference between the left-hand side and the right-hand side of (5.9) is computed by

$$\frac{1}{2}\big\langle \hat{z} \cup (\hat{z} - \hat{\lambda}^c), [T] \big\rangle \tag{5.10}$$

This pairing is well-defined, because $\hat{z}$ is compactly supported. Moreover, it is an integer, because $\hat{\lambda}^c$ is cocycle representative of an integral lift of the Wu class. We deduce that modulo 1, the two sides of (5.9) coincide. $\square$

Theorem 5.7. *$\mathcal{Q}^c$ is compatible with Q^c, in the sense of Definition 2.5.*

Proof. Let $y_\phi \in T^{2\ell+1}(M_\phi) \subset H^{2\ell+2}_{\rm tors}(M_\phi, \mathbb{Z})$ of order k and assume that y_ϕ comes from a class $x \in H^{2\ell+1}(M, \mathbb{Z})$ satisfying $kx = (1 - \phi^*)v$. Let $y_1 = x \cup t \in H^{2\ell+2}(M \times S^1, \mathbb{Z})$. Let $[W_\phi, z_\phi, \check{\mu}^c_\phi, \check{\lambda}^c_\phi]$ be a bordism trivialization of $[M_\phi, y_\phi, \check{\mu}^c_\phi]$ and $[W_1, z_1, \check{\mu}^c_1, \check{\lambda}^c_1]$ be a bordism trivialization of $[M_1, y_1, \check{\mu}^c_1]$. We have

$$\begin{aligned}
\mathcal{Q}^c(y_\phi) &= -\frac{1}{2}\big(B^*(z_\phi, z_\phi) - B^*(z_\phi, \lambda^c_\phi)\big) \\
&= -\frac{1}{2}\left(\big\langle \hat{z}_\phi \cup (\hat{z}_\phi - \hat{\lambda}^c_\phi), [W_\phi, \partial W_\phi]\big\rangle - \frac{1}{k}\big\langle \hat{y}_\phi \cup \hat{u}_\partial, [M_\phi]\big\rangle\right) \\
&= -\frac{1}{2}\left(\big\langle \hat{z}_1 \cup (\hat{z}_1 - \hat{\lambda}^c_1), [W_1, \partial W_1]\big\rangle - \frac{1}{k}\big\langle x \cup v, [M]\big\rangle\right) \\
&= -Q^c(x) + \frac{1}{2k}\big\langle x \cup u, [M]\big\rangle \\
&= \mathcal{Q}_T(y_\phi)
\end{aligned} \tag{5.11}$$

where all the equalities are understood modulo 1. To go from the first line to the second, we just used the expression (2.21) for B^*, with $\hat{u}_\partial$ a cocycle satisfying $k\hat{y}_\phi = d\hat{u}_\partial$. On the third

line, we expressed the first term in terms of data on the trivial mapping torus using Lemma 5.6, and used the equality proved in Lemma 5.9 below. On the fourth line we used (5.7) on the first term, and finally used the definition (2.16) of $\mathcal{Q}_T$ on the last line.

Moreover, $\mathcal{Q}^c$ vanishes on $H^{2\ell+1}_\phi(M,\mathbb{Z}) \subset H^{2\ell+2}(M_\phi,\mathbb{Z})$. Therefore it satisfies Definition (2.5). ☐

As we chose $\check{\lambda}^c$ so that $\mathcal{Q}^c$ vanishes on $H^{2\ell+2}_\phi(M,\mathbb{Z}) \subset H^{2\ell+2}(M_\phi,\mathbb{Z})$, the following corollary is immediate.

Corollary 5.8. *$\mathcal{Q}^c$ coincides with the quadratic refinement $\mathcal{Q}_{Q^c}$ of L_{M_ϕ} induced by Q^c in the sense of Definition 2.6.*

The following lemma is necessary to complete the proof of Theorem 5.7.

Lemma 5.9. *Using the same notations as in the proof of Theorem 5.7, we have*

$$\left\langle \hat{y}_\phi \cup \hat{u}_\partial, [M_\phi] \right\rangle = \left\langle x \cup v, [M] \right\rangle . \tag{5.12}$$

Proof. We will construct explicitly the cocycle $\hat{u}_\partial$. Given cocycles representatives $\hat{x}$ and $\hat{v}$, the relation satisfied by x and v implies that there exists $\hat{r} \in C^{2\ell+1}(M)$ such that

$$d\hat{r} = k\hat{x} - (1-\phi^*)\hat{v} \ . \tag{5.13}$$

Let $\hat{t}$ be a bump cocycle representing the pull back t of the generator of $H^1(S^1,\mathbb{Z})$, supported on an interval I. Let R_α be the transformation of the mapping torus corresponding to a rotation of an angle α of the base. R^*_α deforms continuously $\phi^*\hat{v} \cup \hat{t}$ to $\hat{v} \cup \hat{t}$ when α runs from 0 to 2π. $(1-\phi^*)\hat{v} \cup \hat{t}$ is therefore trivial in cohomology. This means that there exists a cochain $\hat{s}$ satisfying

$$d\hat{s} = (1-\phi^*)\hat{v} \cup \hat{t} \ . \tag{5.14}$$

Therefore we obtain a cochain $\hat{u}_\partial = \hat{r} \cup \hat{t} + \hat{s}$ such that $d\hat{u}_\partial = k\hat{y}_\phi$.

If I is an interval disjoint from the support of $\hat{t}$, $\hat{s}|_{M\times I} = \hat{v}$. Let $\hat{t}_1$ be a cocycle generating $H^1(S^1,\mathbb{Z})$ having support in I. To compute $\left\langle \hat{y}_\phi \cup \hat{u}_\partial, [M_\phi] \right\rangle$, we represent $\hat{y}_\phi$ by the cocycle $\hat{x} \cup \hat{t}_1$ and we obtain (5.12). ☐

6 An example

We take $M = S^3 \times S^3$, a product of three-spheres. M is spin and admits an integral lift given by minus half the first Pontryagin class $\nu = -p_1/2$. A Riemannian metric on M provides a

functorial lift. We have $H^3(M,\mathbb{Z}) = \mathbb{Z}^2$. The two generators v_1 and v_2 are the top classes on one sphere extended as constants on the other. They form a Darboux basis. As shown in the appendix of [1], any element of $\mathrm{Sp}(2,\mathbb{Z})$ acting on $H^3(M,\mathbb{Z})$ can be realized as the pull-back action of a diffeomorphism of M.

Let us compute the quadratic refinement Q^c. Using our Darboux basis, a useful parameterization of the quadratic refinements of L_M is given by a vector $\eta \in \frac{1}{2}H^3(M,\mathbb{Z})/H^3(M,\mathbb{Z})$:

$$Q^c(x = x_1 + x_2) = \frac{1}{2}L_M(x_1,x_2) + L_M(\eta,x) \mod 1 \ , \tag{6.1}$$

where we decomposed x into its components x_1 and x_2 on the Darboux basis. To determine Q^c, it is sufficient to determine its value on v_1 and v_2. We consider the trivial mapping torus $\hat{M} = M \times S_1$ and endow it with the class $v_1 \cup t$. We can take $W = S^3 \times B^4 \times S^1$, where B^4 is a four dimensional ball filling the three-sphere along which v_1 is a constant 0-cocycle. $v_1 \cup t$ extends to a class z in $H^4(W,\mathbb{Z})$ by extending the constant 0-cocycle on the three-sphere to the four-ball. It is clear that $z \cup z = 0$. As the tangent bundle of W is trivial, we have $\nu = -p_1/2 = 0$. From (5.7), we deduce that $Q^c(v_1) = 0$. A symmetric argument implies that $Q^c(v_2) = 0$. Q^c is therefore the quadratic refinement corresponding to $\eta = (0,0)$. The topological invariant $A(Q^c)$ is equal to zero.

Remark 6.1. As was explained in Section 5.3, the canonical quadratic refinement Q^c is a topological invariant of the manifold M. One might therefore have thought that it should be invariant under the full group of diffeomorphisms $\mathcal{D}$ of M. But in view of the example above, this is manifestly not true. Indeed, as we already remarked, the mapping class group of $S^3 \times S^3$ is $\mathrm{Sp}(2,\mathbb{Z})$, and its action on quadratic refinements is affine (see for instance equation (4.9) in [1]). It leaves $\eta = (1,1)$ invariant but permutes the three other quadratic refinements.

The reason for the lack of invariance is the following. To define the quadratic refinement, we used one manifold W_x bounded by the trivial mapping torus of M for each $x \in H^{2\ell+2}_{\mathrm{free}}(M,\mathbb{Z})\otimes\mathbb{Z}_2$. Denote by $\mathcal{D}_x$ the subgroup of $\mathcal{D}$ consisting of diffeomorphisms of M that can be realized as restrictions of diffeomorphisms of W_x. The quadratic refinement is invariant under $\mathcal{D}_Q := \bigcap_x \mathcal{D}_x$, but not necessarily under $\mathcal{D}$. Of course if some other choice of family $\{W_x\}$ leads to a different group $\mathcal{D}'_Q \subset \mathcal{D}$, the quadratic refinement will be invariant under the minimal subgroup of $\mathcal{D}$ containing both $\mathcal{D}_Q$ and $\mathcal{D}'_Q$, but there is no reason why this should be all of $\mathcal{D}$.

Let us now turn to a non-trivial mapping torus. Consider a diffeomorphism ϕ inducing a T^2 transformation of $Sp(2,\mathbb{Z})$ on $H^3(M,\mathbb{Z})$, that is $x \to x + 2\langle x \cup v_1, [M]\rangle v_1$. This transformation

preserves the quadratic refinement Q^c. A diffeomorphism ϕ realizing this transformation maps the first three-sphere in $S^3 \times S^3$ to a sphere in the same homotopy class. Let M_ϕ be the corresponding mapping torus. By filling the first three-sphere with a four-ball B^4, we obtain an 8-dimensional manifold W_ϕ bounded by M_ϕ. $H^4(W_\phi, \hat{M}_\phi, \mathbb{Z})$ is one dimensional, generated by the top class v_{B_4} of B^4. We have $v_{B_4} \cup v_{B_4} = 0$ so $\sigma_{W_\phi} = 0$.

On the other hand, the Arf invariant of the quadratic refinement $\mathcal{Q}^c$ on the torsion cohomology of M_ϕ depends only on the action of ϕ on the cohomology of M, here the element $T^2 \in Sp(2, \mathbb{Z})$. See Section 3.4 of [2] for details about how to compute it. Table 1 in this paper shows that $A(\mathcal{Q}^c) = 1/8 \mod 1$. Using (4.11), we conclude that

$$\int_W \underline{\lambda}^c \wedge \underline{\lambda}^c = 1 \mod 8 \; . \tag{6.2}$$

This result should have some importance in the context of anomaly cancellation in six dimensional quantum field theories, because it shows that neither $A(\mathcal{Q}^c)$ nor $\int_W \underline{\lambda}^c \wedge \underline{\lambda}^c$ have special evenness properties.

7 Application to the global anomaly formula

Our motivation to study quadratic refinements comes from a formula for the global gravitational anomaly of the self-dual field theory derived recently in [1, 2]. We refer the reader to the introductions of these two papers for a background on gravitational anomalies and their interest for the study of quantum gravity theories, such as string or M-theory. In this section, we present the mathematical framework underlying this global anomaly formula and explain the use of the results above, especially of Theorem 4.11. This section is purely motivational and we do not attempt to be completely rigorous.

7.1 Line bundle over the space of metrics modulo diffeomorphisms

Let $\mathcal{M}$ be the space of Riemannian metrics on a $4\ell + 2$-dimensional compact oriented manifold M. Let $\mathcal{D}$ the group of diffeomorphisms leaving fixed a point and its tangent space. Given a quadratic refinement Q of the pairing L_M on $H^{2\ell+1}(M, \mathbb{Z}) \otimes \mathbb{Z}_2$, let $\mathcal{D}_Q$ be the subgroup of $\mathcal{D}$ leaving Q fixed. The quotient $\mathcal{M}/\mathcal{D}_Q$ is a smooth infinite dimensional manifold. There is a fiber bundle $\mathcal{E} := (M \times \mathcal{M})/\mathcal{D}_Q$ with fiber M over $\mathcal{M}/\mathcal{D}_Q$. Each fiber is equipped with the Riemannian metric given by the projection map. Remark that any loop c on $\mathcal{M}/\mathcal{D}_Q$ defines a mapping torus $\hat{M}_c := \mathcal{E}|_c$ of M.

In [1], we studied the anomaly bundle of the self-dual field, which is the line bundle over $\mathcal{M}/\mathcal{D}_Q$ of which the partition function of the self-dual field is a section. We describe it next as briefly as possible (see [1] for details). Consider the Dirac operator D coupled to chiral spinors on M. D can be seen as $d + d^\dagger$, where $d^\dagger$ is the codifferential $- * d*$, restricted to the space of self-dual forms. [8] The index theory for families of Dirac operators associates to D and $\mathcal{E}$ a line bundle $\mathscr{D}$ over $\mathcal{M}/\mathcal{D}_Q$, the determinant bundle of the family of Dirac operators. $\mathscr{D}$ comes equipped with a natural connection, the Bismut-Freed connection [19, 20].

Another way of constructing line bundles over $\mathcal{M}/\mathcal{D}_Q$ is as follow. The Hodge star operator associated to a given Riemannian metric on M defines a complex structure on the symplectic space $H^{2\ell+1}(M,\mathbb{R})$. Therefore, we get a map from $\mathcal{M}$ to the Siegel upper half-space $\mathcal{C}$ parameterizing the complex structures of $H^{2\ell+1}(M,\mathbb{R})$. Quotienting by the action of $\mathcal{D}_Q$ on both sides, we get a map from $\mathcal{M}/\mathcal{D}_Q$ to $\mathcal{C}/\Gamma_Q$, where Γ_Q is a subgroup of $\mathrm{Sp}(2b_{2\ell+1}(M),\mathbb{Z})$, with b_n the nth Betti number. Line bundles on the modular variety $\mathcal{C}/\Gamma_Q$ can therefore be pulled back to $\mathcal{M}/\mathcal{D}_Q$. We call $\mathscr{K}$ the pull back of the determinant of the Hodge bundle over $\mathcal{M}/\mathcal{D}_Q$, and we have $\mathscr{K} \simeq \mathscr{D}^{-1}$ as topological bundles. Associated to the quadratic refinement Q is a theta function on $H^{2\ell+1}(M,\mathbb{R}) \times \mathcal{C}$, restricting to a "theta constant" on $\mathcal{C}$. The theta constant is the pull-back of the section of a non-trivial line bundle over $\mathcal{C}/\Gamma_Q$, the theta bundle $\mathscr{C}_Q$. Define the flat bundle $\mathscr{F}_Q := (\mathscr{C}_Q)^2 \otimes \mathscr{K}^{-1}$. The square of the anomaly bundle $\mathscr{A}_Q$, as a bundle with connection, is

$$(\mathscr{A}_Q)^2 = \mathscr{D} \otimes \mathscr{F}_Q \ . \tag{7.1}$$

The anomaly bundle itself is topologically isomorphic to $\mathscr{C}_Q$, and equipped with the connection making (7.1) an equality of bundles with connections. An explicit expression for the connection form was derived in [21].

The problem of determining the global gravitational anomaly of the self-dual field theory amounts to computing the holonomies of the connection on $\mathscr{A}_Q$ along loops in $\mathcal{M}/\mathcal{D}_Q$. In [2], we showed, although not completely rigorously, that it can be done as follows. Let c be a loop in $\mathcal{M}/\mathcal{D}_Q$. Restricting $\mathcal{E}$ to c, we get a mapping torus M_c. As we saw, the quadratic refinement Q defines a quadratic refinement $\mathcal{Q}_Q$ of the linking pairing L_{M_c} on $H^{2\ell+2}_{\mathrm{tors}}(M_c,\mathbb{Z})$. Let W be a manifold bounded by M_c and let $\underline{\lambda}_Q$ be a relative lift of the Wu class compatible with $\mathcal{Q}_Q$ in the sense of Definition 2.8. Then the holonomy of the connection along c is given by

$$\frac{1}{2\pi i} \ln \mathrm{hol}_{\mathscr{A}_Q}(c) = \frac{1}{8} \int_W \left(\underline{\lambda}_Q \wedge \underline{\lambda}_Q - \underline{L} \right) , \tag{7.2}$$

[8] We define the space of self-dual forms to be the image of the action of $1 - i^p *$ on $\Omega^p(M)$, $p \leqslant 2\ell + 1$.

where $\underline{L}$ denotes the Hirzebruch L-genus of W. In (7.2), the second term is metric-dependent. We take a limit on the metric on M_c such that the length of the base circle goes to infinity, and a compatible metric on W.

Main result for physical applications The problem when trying to use (7.2) is that $\underline{\lambda}_Q$ is not explicit. The main insight we gain from the mathematical considerations of the previous sections (Theorem 4.11 and Corollary 5.8), is that if we are considering a family of manifolds admitting a functorial lift of their Wu class, and if $Q = Q^c$, as defined in (5.7), then $\underline{\lambda}_Q = \underline{\lambda}^c$.

In dimension 2 and 10, $\underline{\lambda}^c$ can be taken to vanish. This allowed to check the cancellation of global gravitational anomalies in "cohomological" type IIB supergravity [2]. In dimension 6, $\underline{\lambda}^c$ cannot vanish. The results in this paper provide a construction for this class, which should prove crucial in order to check anomaly cancellation in six dimensional supergravities and in the world-volume theories of the five-branes.

Let us also remark that the requirement we made in [2] that the torsion cohomology of W should restrict trivially on M_c is not necessary in case $Q = Q^c$. The aim of this restriction was to ensure that the element $b \in H^{2\ell+2}_{\rm tors}(\partial W, \mathbb{Z})$ defined in (2.26) vanishes. But as noted in the proof of Theorem 4.11, b vanishes automatically for $Q = Q^c$.

7.2 Relation to the original construction

The partition function of the self-dual field theory involves a theta function, whose characteristic needs to be specified. More invariantly, we need to specify a holomorphic line bundle over the intermediate Jacobian $\mathcal{J} = H^{2\ell+1}(M, \mathbb{R})/H^{2\ell+1}_{\rm free}(M, \mathbb{Z})$ whose first Chern class coincides with the pairing $\int_M \bullet \wedge \bullet \in \bigwedge^2 \left(H^{2\ell+1}(M, \mathbb{R})\right)^* \simeq \bigwedge^2 T^*(\mathcal{J})$. Symmetric line bundles are invariant under the antipodal map of the torus $\mathcal{J}$. They are in bijection with quadratic refinements of the pairing L_M on $H^{2\ell+1}_{\rm free}(M, \mathbb{Z}) \otimes \mathbb{Z}_2$. In the previous sections, we made the choice of a symmetric line bundle on $\mathcal{J}$ by choosing a quadratic refinement Q. This was also the approach adopted in [22, 1, 2].

However, the case of the M5-brane [3, 17, 23] is more subtle. First, the holomorphic line bundle has to be defined on the *shifted* intermediate Jacobian $\mathcal{J}_\nu$. Recall that a shifted differential cocycle is a differential cochain $\check{C}$ such that $d\check{C} = (0, \hat{\nu}/2, 0)$, where $\hat{\nu}$ is the real cocycle associated to the form lift $\underline{\nu}$. $\mathcal{J}_\nu$ can be pictured as the set of shifted differential cohomology classes of form $\check{C}_0 + (0, \hat{x}, 0)$, with $\check{C}_0 = (0, 0, -\underline{\nu}/2)$. Second, the line bundle should depend continuously on the Riemannian metric of M [23]. This continuous dependence on the metric was also present in the original proposal for the line bundle on $\mathcal{J}_\nu$ associated to the five-brane

[3]. In this section, we compare the choice of line bundle in this reference to our canonical choice.

The data of the relevant holomorphic line bundle over $\mathcal{A}$ is contained in the set of holonomies along linear paths in $\mathcal{J}_\nu$. In [3], the holonomy along a path of shifted differential cocycles $\check{C}_x(s) := \check{C}_0 + (0, s\hat{x}, 0)$, where $\hat{x}$ a real cocycle corresponding to an element of $H^{2\ell+1}_{\rm free}(M, \mathbb{Z})$, was defined as

$$\mathrm{hol_W}(\check{C}_x) = \frac{1}{8}\int_W (\underline{\nu} + 2\underline{z})^2 - \underline{\nu}^2 = \frac{1}{2}\int_W \underline{z} \wedge (\underline{\nu} + \underline{z}) \mod 1 , \tag{7.3}$$

where W is again a manifold bounded by the trivial mapping torus $M \times S^1$ and $\underline{z}$ is a form extending $\underline{x} \wedge ds$. Using the relation $\underline{\lambda}^c = \underline{\nu} - 2\underline{\mu}^c$, we get

$$\mathrm{hol_W}(\check{C}_x) = \frac{1}{2}\int_W \underline{z} \wedge (\underline{z} + \underline{\lambda}^c) + \int_W \underline{z} \wedge \underline{\mu}^c \mod 1 . \tag{7.4}$$

The first term is $Q^c(x)$, pulled back to the shifted Jacobian by the linear map sending $\check{C}_0$ to $(0,0,0)$. The second term depends on the Riemannian metric on ∂W, because neither $\underline{z}$ nor $\underline{\mu}^c$ vanish on the boundary. Hence it depends on a choice of Riemannian metric on M. The holonomy formula (7.2) will be studied in this more complex case in future work.

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A Some results in cobordism theory

Spin manifolds admits functorial integral lifts of the Wu class and form an important class of examples where the formalism developed in the main body of this paper can be applied. The latter however requires that manifolds of dimension $4\ell + 3$ endowed with two classes of degree $2\ell + 2$ can always be seen as boundaries of manifolds on which the two classes extend. In this appendix, we show that this is the case for $\ell = 0, 2$ and ℓ odd, and that the obstruction is non-vanishing for other values of ℓ. Let us recall that as far as physical applications are concerned, only the cases $\ell = 0, 1$ and 2 are of interest.

A.1 Background material about spectra[9]

Recall that a spectrum K is a sequence of (pointed) CW complexes K_n together with maps $\Sigma K_n \to K_{n+1}$. The homotopy groups of a spectrum are defined by $\pi_i(K) = \lim_{\to} \pi_{i+n}(K_n)$. For our purpose, a map between of degree m between a spectrum K and a spectrum L is a sequence of maps $K_n \to L_{n-m}$ commuting with the suspension maps defining the two spectra. This sequence of maps only needs to be defined starting from an arbitrary positive integer N. Considering suitable equivalence classes of these maps, the set of spectra forms a category. It is also possible to define a smash product $\wedge$ on this category.

Given any CW complex X, we can construct the corresponding suspension spectrum $\Sigma^\infty X$, by defining $(\Sigma^\infty X)_n = \Sigma^n X$. In particular, we have the sphere spectrum S, for which S_n is the sphere of dimension n.

The interest of spectra is that they are naturally associated to generalized (co)homology theories. Given a spectrum K, we can define a reduced generalized homology theory

$$\tilde{K}_n(X) = [S, K \wedge X]_n$$

where $[\ ,\ ]_n$ denotes homotopy classes of maps of degree n. With this definition, X can be either an ordinary CW complex or a spectrum. If X is a CW complex, it can be interchanged with its suspension spectrum $\Sigma^\infty X$ in the formula above.

The integral homology spectrum $H\mathbb{Z}$ is the spectrum having $(H\mathbb{Z})_n = K(\mathbb{Z}, n)$.

A.2 Spin cobordism group of a point

It is useful to think about our extension problem in a sequential way. The first potential obstruction arises when trying to promote a spin manifold of dimension $4\ell+3$ to the boundary of a spin manifold of dimension $4\ell+4$. The obstruction is given by the degree $4\ell+3$ spin bordism group of a point $\Omega^{\rm spin}_{4\ell+3} := \Omega^{\rm spin}_{4\ell+3}(\rm pt)$, classifying bordism classes of maps of manifolds of dimension $4\ell+3$ to a point. $\Omega^{\rm spin}_\bullet$ is a generalized homology, with spectrum $MSpin$. $MSpin_n$ can be seen as the Thom space of the universal bundle of dimension n over the classifying space $BSpin$.

Lemma A.1. $\Omega^{\rm spin}_{4\ell+3}$ *vanishes for* $\ell = 0, 2$ *and for* ℓ *odd.*

Proof. This is a straightforward consequence of Theorem 2.2 in [25]. Indeed, there is a system of generators for $\pi_\bullet(MSpin) \simeq \Omega^{\rm spin}_\bullet$, which can be described as follows. Given a sequence of

[9] Details can be found in the book [24].

integers $J = (j_1, ..., j_k)$ such that $k \geqslant 0$ and $j_i > 1$, define $n(J) = \sum_{i=0}^{k} j_i$. Define also $p(n)$ to be equal to 1 for positive n if $n = 0, 1, 2, 4$ mod 8 and zero else, i.e. $p(n)$ is the rank of $KO^n(\mathrm{pt})$ over $\mathbb{Z}_2$. In addition to generators in degree 0 mod 8, for each J, there are $p(d-s)$ generators in degree d, where the shift s is equal to $4n(J)$ if $n(J)$ is even, and to $4n(J)-2$ if $n(J)$ is odd.

It is clear that the shift s is always equal either to 0 modulo 8 or to 2 modulo 8. From the definition of p, we see that odd degree generators are present only in degree 1 or 3 modulo 8. As a result, $\Omega^{\mathrm{spin}}_{4\ell+3}$ always vanishes for ℓ odd. Inspection of low degree cases show in addition that it vanishes for $\ell = 0$ and 2. □

A.3 Extending a single cohomology class

We now go one step further in the extension problem and try to extend a single class of degree $2\ell+1$ from the manifold of dimension $4\ell+3$ to the bounded manifold. General arguments[10] show that the reduced spin bordism classes of maps to a pointed space X are given by the stable homotopy groups of $MSpin \wedge \Sigma^\infty X$:

$$\tilde{\Omega}^{\mathrm{spin}}_n\big(X\big) := [S, MSpin \wedge \Sigma^\infty X]_n \ . \tag{A.1}$$

"Reduced" is used here in the sense of reduced homology: the full set of bordism classes of maps is given by $\Omega^{\mathrm{spin}}_n(X) = \tilde{\Omega}^{\mathrm{spin}}_n(X) \oplus \Omega^{\mathrm{spin}}_n(\mathrm{pt})$.

Recall that an integral cohomology class of degree $2\ell+2$ on a manifold $\hat{M}$ can be classified by a homotopy class of maps in $[\hat{M}, K(\mathbb{Z}, 2\ell+2)]$. Therefore bordism classes of manifolds of dimension $4\ell+3$ endowed with a class of degree $2\ell+1$ are classified by

$$\tilde{\Omega}^{\mathrm{spin}}_{4\ell+3}\big(K(\mathbb{Z}, 2\ell+2)\big) := [S, MSpin \wedge \Sigma^\infty K(\mathbb{Z}, 2\ell+2)]_{4\ell+3} \ , \tag{A.2}$$

provided $\Omega^{\mathrm{spin}}_{4\ell+3}(\mathrm{pt}) = 0$.

Lemma A.2. $\tilde{\Omega}^{\mathrm{spin}}_{4\ell+3}\big(K(\mathbb{Z}, 2\ell+2)\big) = 0$

Proof. This proof follows the arguments in Section 2 of [27]. By definition, $(\Sigma^\infty K(\mathbb{Z}, 2\ell+2))_n = \Sigma^n K(\mathbb{Z}, 2\ell+2)$. There is a map of spectra

$$\Sigma^\infty K(\mathbb{Z}, 2\ell+2) \to \Sigma^{2\ell+2} H\mathbb{Z} \tag{A.3}$$

induced by the homotopy equivalences $\Sigma^n K(\mathbb{Z}, 2\ell+2) \simeq K(\mathbb{Z}, 2\ell+2+n)$. As $\Sigma^{2\ell+2} H\mathbb{Z}$ is $2\ell+1$-connected, the map (A.3) induces, via Freudenthal's theorem, an $4\ell+4$-equivalence. This

[10] See for instance [26] for a particularly clear presentation.

implies that we have an isomorphism

$$
\begin{aligned}
\Omega^{\text{spin}}_{4\ell+3}(K(\mathbb{Z},2\ell+2)) &:= [S, MSpin \wedge \Sigma^\infty K(\mathbb{Z},2\ell+2)]_{4\ell+3} \\
&\simeq \left[S, MSpin \wedge \Sigma^{2\ell+2} H\mathbb{Z}\right]_{4\ell+3} \\
&\simeq [S, MSpin \wedge H\mathbb{Z}]_{2\ell+1} \\
&\simeq [S, H\mathbb{Z} \wedge MSpin]_{2\ell+1} \\
&= H_{2\ell+1}(MSpin, \mathbb{Z}) \ ,
\end{aligned}
\tag{A.4}
$$

using the commutativity of the smash product. By the Thom isomorphism, $H_{2\ell+1}(MSpin,\mathbb{Z}) \simeq H_{2\ell+1}(BSpin,\mathbb{Z})$. To compute the homology of $BSpin$, we use the following facts.

- The pull back map π^* induced from the fibration $BSpin \to BSO$ is an isomorphism on cohomology with value in a ring containing 1/2, and surjective on cohomology with $\mathbb{Z}_2$ coefficients (pages 290-292 of [28]).
- All the torsion in $H^\bullet(BSO,\mathbb{Z})$ has order 2 [29]. The previous fact implies that all the torsion in $H^\bullet(BSpin,\mathbb{Z})$ has order 2 as well.
- The stable cohomology with $\mathbb{Z}$ coefficients of BSO has no torsion in even degrees [29].

Consider now the following commutative square for p odd.

$$
\begin{tikzcd}
H^p(BSO,\mathbb{Z}_2) \arrow[r, "\delta"] \arrow[d, "\pi^*"] & H^{p+1}(BSO,\mathbb{Z}) \arrow[d, "\pi^*"] \\
H^p(BSpin,\mathbb{Z}_2) \arrow[r, "\delta"] & H^{p+1}(BSpin,\mathbb{Z})
\end{tikzcd} \ .
\tag{A.5}
$$

As $H^{p+1}(BSO,\mathbb{Z})$ is torsion-free, the top δ map vanishes. This implies that $H^{p+1}(BSpin,\mathbb{Z})$ has no torsion of order two, hence no torsion at all. Using the universal coefficient theorem, this implies that $H_p(BSpin,\mathbb{Z})$ for odd p is torsion free, hence vanishes. $\square$

A.4 Extending a pair of cohomology classes

We now consider the problem of extending a pair of cohomology classes of degree $2\ell+2$, which is the extension property we use several times in our arguments. Clearly, pair of such classes are classified by homotopy classes of maps in $[\hat{M}, K(\mathbb{Z},2\ell+2)\times K(\mathbb{Z},2\ell+2)]$ provided $\Omega^{\text{spin}}_n(\text{pt}) = 0$. Given two CW complexes A and B, the direct product $A\times B$ can be reexpressed in terms of the smash product as $A_+ \wedge B_+$, where A_+ denotes the disjoint union of A and a point. Moreover,

for any generalized homology theory, E, we have $\tilde{E}(A_+) = E(A) = \tilde{E}(A) \oplus E(\mathrm{pt})$, where as before, $\tilde{E}$ denotes the reduced cohomology.

Using these facts, it is easy to see that provided $\Omega^{\mathrm{spin}}_{4\ell+3}(\mathrm{pt})$ and $\Omega^{\mathrm{spin}}_{4\ell+3}(K(\mathbb{Z}, 2\ell+2))$ both vanish, the vanishing of $\Omega^{\mathrm{spin}}_{4\ell+3}(K(\mathbb{Z}, 2\ell+2) \times K(\mathbb{Z}, 2\ell+2))$ is equivalent to the vanishing of $\Omega^{\mathrm{spin}}_{4\ell+3}(K(\mathbb{Z}, 2\ell+2) \wedge K(\mathbb{Z}, 2\ell+2))$.

But $\pi_n(K(\mathbb{Z}, 2\ell+2) \wedge K(\mathbb{Z}, 2\ell+2))$ vanishes for $n < 4\ell+4$, so

$$\begin{aligned}\Omega^{\mathrm{spin}}_{4\ell+3}(K(\mathbb{Z}, 2\ell+2) \wedge K(\mathbb{Z}, 2\ell+2)) &:= \\ &\big[S, MSpin \wedge \Sigma^\infty K(\mathbb{Z}, 2\ell+2) \wedge \Sigma^\infty K(\mathbb{Z}, 2\ell+2)\big]_{4\ell+3} = 0\ . \end{aligned} \tag{A.6}$$

Combining the above with Lemmas A.1 and A.2, we have:

Proposition A.3. *For ℓ odd or $\ell = 0, 2$, given a spin manifold $\hat{M}$ of dimension $4\ell+3$ and two cohomology classes of degree $2\ell+2$, it is always possible to find a spin manifold W bounded by $\hat{M}$ on which both classes extend. For ℓ even different from 0 or 2, the extension is possible provided $\hat{M}$ is a boundary.*

Let us remark that equation (A.6) generalizes to

$$\big[S, X \wedge \Sigma^\infty K(\mathbb{Z}, 2\ell+2) \wedge \Sigma^\infty K(\mathbb{Z}, 2\ell+2)\big]_{4\ell+3} = 0\ . \tag{A.7}$$

for any spectrum X associated to a generalized homology theory. This implies that for any category of manifold admitting an integral lift of the Wu class of degree $2\ell+2$, if it is possible to solve the first two extension problems (i.e. find a bounded manifold, and extend a single class of degree $2\ell+2$), it is always possible to solve the third one (i.e. extend a pair of classes). Moreover, the same reasoning shows that the obstruction to extending a higher number of classes vanish as well. This fact is used in the proof of Proposition 5.5, where we need to extend three classes.

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